

Clemens-Schmid exact sequence in characteristic p

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Abstract

For a semistable family of varieties over a curve in characteristic p , we prove the existence of a "Clemens-Schmid type" long exact sequence for the p -adic cohomology. The cohomology groups appearing in such a long exact sequence are defined locally.

1 Introduction

Let Δ denote an open disk around 0 in the complex plane. Let X be a smooth complex variety which is a Kähler manifold. Consider a semi-stable degeneration $\pi : X \rightarrow \Delta$, i.e., a holomorphic, proper and flat map of relative dimension n such that π is smooth outside $X_0 = \pi^{-1}(0)$, such that the fiber X_0 is a divisor with global normal crossings (in other words $X_0 = \sum X_{0,i}$ is a sum of irreducible components $X_{0,i}$ of X_0 meeting transversally and each $X_{0,i}$ is smooth). In this situation, for any m , one can associate a limit cohomology H_{\lim}^m for X_0 (see [MO84] or [ST76]). This H_{\lim}^m is endowed with a nilpotent monodromy operator N and a weight filtration from a mixed Hodge structure. One has $H^m(X_t) \simeq H_{\lim}^m$ as vector spaces for $t \neq 0$ where $X_t = \pi^{-1}(t)$; moreover, a topological argument shows that $H^m(X) \simeq H^m(X_0) := H^m$ and $H_m(X) \simeq H_m(X_0) := H_m$ as well. By i we will indicate the inclusion $X_t \rightarrow X$. Then it is possible to define the Clemens-Schmid exact sequence (respecting MHS) [CL77] Chap. 1, 3.7 (see also [MO84]):

$$\cdots \rightarrow H_{2n+2-m} \xrightarrow{\alpha} H^m \xrightarrow{i^*} H_{\lim}^m \xrightarrow{N} H_{\lim}^m \xrightarrow{\beta} H_{2n-m} \xrightarrow{\alpha} H^{m+2} \rightarrow \cdots,$$

where the maps α are the natural maps arising from the Poincaré duality for X_0 considered as closed in the smooth variety X , and the maps β are again obtained via Poincaré duality for $H^m(X_t) \simeq H_{2n-m}(X_t)$ and then composed with the natural map $H_{2n-m}(X_t) \rightarrow H_{2n-m}(X)$ dual to i^* .

In order to prove the exactness of this long sequence one needs more than a topological argument which connects a global definition of the cohomology of X with support in X_0 to a sequence involving the cohomology of the special and generic fibers. Indeed one also needs a "Weights" argument. In fact one has to use the fact that the sequence respects the weight filtrations of mixed Hodge structures of the vector spaces involved and moreover that the weight and monodromy filtrations on H_{\lim}^n coincide. We recall, also, that the structure of the limit cohomology has been considered in the framework of log-geometry (see [IL94]).

In this article we deal with the analogous situation in characteristic $p > 0$. Namely, we consider the following morphism

$$f : X \rightarrow C$$

over a finite field k of characteristic p , where X is a smooth variety of dimension $n+1$, C is a smooth curve and f is a proper and flat morphism. We suppose that for a k -rational point s of C , the fiber at s of f , X_s ,

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is a normal crossing divisor (NCD for simplicity) and f is smooth outside X_s . We prove the existence of a Clemens-Schmid sequence in this situation. For simplicity, we will indicate by \mathcal{V} a complete and absolutely unramified DVR whose residue (resp. fraction) field is k (resp. K of characteristic 0):

(1)

$$\begin{aligned} \cdots &\rightarrow H_{\text{rig}}^m(X_s) \xrightarrow{\gamma} H_{\text{log-crys}}^m((X_s, M_s)/\mathcal{V}^\times) \otimes K \xrightarrow{N_m} H_{\text{log-crys}}^m((X_s, M_s)/\mathcal{V}^\times) \otimes K(-1) \xrightarrow{\delta} H_{X_s, \text{rig}}^{m+2}(X) \\ &\xrightarrow{\alpha} H_{\text{rig}}^{m+2}(X_s) \rightarrow \cdots, \end{aligned}$$

where (a) means the a -th Tate twist of Frobenius structure.

The role of the limit cohomology will be played by the log-crystalline cohomology of the log-scheme X_s endowed with the log-structure induced by the log-structure of X given by the NCD X_s itself. We denote this limit cohomology by $H_{\text{log-crys}}^m((X_s, M_s)/\mathcal{V}^\times)$ (\mathcal{V}^\times is the log-structure on \mathcal{V} associated to $1 \mapsto 0$). We will then consider the cohomology of the special fiber X_s without any structure: and here we will apply rigid cohomology, $H_{\text{rig}}^m(X_s)$. We now need to replace the "transcendental" topological argument used to construct such a sequence. The underlying idea is that the bridge between the local and the global will be given by two different definitions of the cohomology of X with support in X_s , $H_{X_s, \text{rig}}^m(X)$. Moreover we will link it to the cohomology of the open complement of X_s in X which will be understood in the framework of a generalized log-convergent cohomology theory introduced by Shiho. Hence we will have the long exact sequence

$$(2) \quad \cdots \rightarrow H_{X_s, \text{rig}}^m(X) \xrightarrow{\alpha} H_{\text{rig}}^m(X_s) \rightarrow H_{\text{log-conv}}^m((X_s, M_s)/\mathcal{V}) \rightarrow \cdots.$$

Furthermore the (absolute) log-convergent cohomology groups, $H_{\text{log-conv}}^m((X_s, M_s)/\mathcal{V})$, will be associated with the limit Hodge structure of the special fiber. In fact we will obtain

(3)

$$\cdots \rightarrow H_{\text{log-conv}}^m((X_s, M_s)/\mathcal{V}) \rightarrow H_{\text{log-crys}}^m((X_s, M_s)/\mathcal{V}^\times) \otimes K \xrightarrow{N_m} H_{\text{log-crys}}^m((X_s, M_s)/\mathcal{V}^\times) \otimes K(-1) \rightarrow \cdots,$$

so that we can merge the two sequences (1) to obtain the p -adic analogue of the Clemens-Schmid sequence [CL77].

For the exactness of the merged sequence we note that the log-crystalline cohomology of X_s admits a weight structure (coming from the Frobenius action) and the existence of a monodromy operator. Following the work of Shiho, we will insert our cohomology into a family. Moreover to such a family we will associate a differential operator having a regular singular point at s and endowed with a Frobenius structure. We will re-interpret the monodromy operator in terms of residue of the differential operator at s . In this differential setting the equivalence between the monodromy and weight filtration (given by Frobenius) has been proved by Crew [CR98] 10.8. On the other hand, we also have Poincaré duality in the rigid setting obtaining

$$H_{X_s, \text{rig}}^{m+2}(X) \cong H_{c, \text{rig}}^{2n+2-m-2}(X_s)^\vee(-\dim X) \cong H_{2n-m}^{\text{rig}}(X_s)(-\dim X)$$

[LS07], [PE03], [Be97]: hence (1) is the complete analogue of the classical Clemens-Schmid exact sequence.

We will see that the cohomology group $H_{X_s, \text{rig}}^m(X)$ is isomorphic to the cohomology of the completion \hat{X} of X along X_s with support in X_s (Remark 4.13). Hence the Clemens-Schmid type sequence exists geometrically on the completion \hat{C} of C along the special point s . In this paper we will prove the exactness of the Clemens-Schmid type sequence when k is a finite field because we would like to avoid the difficulty of building up the relative theory. We believe, however, that it is possible to remove the finiteness hypothesis. We do not treat the problem of the exactness for a proper semistable family defined over $\text{Spec } k[[t]]$.

It is tempting to try to see our procedure along the lines of Levine's article on motivic tubular neighborhoods [LE07]. In that article topological methods were replaced by the notion of tubular neighborhoods. In our p -adic realization, this corresponds to the use of the tubes in characteristic 0 for our varieties in characteristic p . With respect to his approach, in our "realization" we have the advantage of

a weight filtration which can be compared to the monodromy filtration and we can prove the exactness of our Clemens-Schmid sequence. We also mention that Nakkajima studied the kernel of the monodromy operator in crystalline settings in [NA06], Sect. 6.

Here is an outline of this paper. After establishing our notations and conventions, we will show in paragraph 3, how Shiho's theory of relative log-cohomology (convergent, analytic and rigid) can be used in our setting. In particular we will understand the log-crystalline cohomology of the special fiber X_s as a fiber at s of the sheaves of relative log-cohomology on C which are endowed with an overconvergent connection, whose residue at s will be the monodromy operator. In the paragraph 4 we will construct the sequence. We will show how the global long exact sequence should be defined using "local" objects via tubes: this will replace the topological methods in the classical case. To do that we will need to compare and link several cohomology theories: these results will be obtained by choosing a good embedding system. In fact, in [SH02] 2.2.4, Shiho was able to define in a functorial way a log tubular neighborhood $]X[_{\mathcal{P}}^{\log}$ from a (not exact, in general) closed immersion $X \rightarrow \mathcal{P}$ of the k -schemes X to the formal \mathcal{V} -scheme \mathcal{P} (under some assumptions). Here we will need to generalize that approach and to construct a good embedding system for étale hypercoverings which admits some exactness properties. This is done in Propositions 4.3 and 4.11. In paragraph 5 we will prove the last ingredient for the exactness of the Clemens-Schmid sequence: the monodromy filtration coincides with the weight filtration for the log-crystalline cohomology of X_s . This will be proved using the theory of the third paragraph: namely the fact that we may view this cohomology as a special fiber at s of a module endowed with a log-connection and a Frobenius structure on the curve (with a log-structure given by the special point) and with monodromy given by the residue of this differential module at that special point (hence the monodromy is unipotent). In this sense we will use Crew's results on the equivalence of two filtrations ([CR98] §10), which, in turn, was an adaptation of Deligne's methods for the étale setting ([DE80] 1.8.4).

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2 Notation and setting

In this paper we will indicate by k a perfect field of characteristic $p > 0$, if not otherwise indicated. For simplicity we denote by \mathcal{V} the ring of Witt vectors of k , K its fraction field. The Frobenius is denoted by σ . We put $\mathcal{V}_i = \mathcal{V}/p^i\mathcal{V}$. Of course one could have taken for \mathcal{V} any complete discrete valuation ring with residue field k and $\text{Frac}(\mathcal{V}) = K$: but all the K -cohomology groups we are going to consider in this case will be defined by tensoring with K the cohomology groups defined over the fraction field of the ring of Witt vectors of k . Hence the ramification does not trouble our constructions and results.

We recall that a divisor $Z \subset Y$ of a Noetherian scheme is said to be a strict normal crossing divisor (SNCD) if Z is a reduced scheme and, if $Z_i, i \in J$ are the irreducible components of Z , then, for any $I \subseteq J$ (which might be empty) the intersection $Z_I = \cap_{i \in I} Z_i$ is a regular scheme of codimension the number of elements of I (or it may be empty). Moreover Y is said to be a normal crossing divisor (NCD) if, étale locally on Y , it is a SNCD.

We consider the following morphism

$$f : X \rightarrow C$$

over a field k , where X is a smooth variety of dimension $n + 1$, C is a smooth curve and f is a proper and flat morphism. We suppose that, for a k -rational point s of C , the fiber at s of f , X_s , is a NCD in X and f is smooth on $X \setminus X_s$. We use (X, M) to denote the scheme X endowed with the log-structure given by the NCD, X_s , while (C, s) denotes the curve C endowed with the log-structure given by s (all for the étale

topology). The induced map $f : (X, M) \rightarrow (C, s)$ is log-smooth and proper. Then s^\times is a log-point given by the k -rational point s of (C, s) , i.e., the induced log-structure given by the closed immersion $s \rightarrow C$ from (C, s) . We refer to such a situation by a cartesian diagram

$$(4) \quad \begin{array}{ccc} (X_s, M_s) & \longrightarrow & (X, M) \\ \downarrow & & \downarrow f \\ s^\times & \longrightarrow & (C, s). \end{array}$$

By \mathcal{V}^\times we indicate \mathcal{V} endowed with the log-structure associated to $\mathbb{N} \ni 1 \mapsto 0 \in \mathcal{V}$. Again \mathcal{V} will indicate \mathcal{V} endowed with the trivial log-structure.

If we say a property is satisfied by a simplicial (formal) scheme (resp. a morphism of simplicial (formal) schemes), then we mean it is satisfied at each level of the simplicial (formal) scheme (resp. the morphism of simplicial (formal) schemes).

By $[A^\bullet \rightarrow B^\bullet]$, we understand the simple complex associated to the double one (also in the simplicial setting) for complexes A^\bullet, B^\bullet .

3 Relative Cohomology

Because in (4), C was a smooth curve over k then it admits a smooth lifting $C_{\mathcal{V}}$ over \mathcal{V} (7.4 III, SGA1): we indicate by $\mathcal{C}_{\mathcal{V}}$ its completion along the special fiber C . Let us fix a lift \hat{s} of s in $C_{\mathcal{V}}$ and a section t as a local coordinate of \hat{s} in $C_{\mathcal{V}}$ over \mathcal{V} . \hat{s} and t also denote a lift of s in $\mathcal{C}_{\mathcal{V}}$ and a local coordinate of \hat{s} in $\mathcal{C}_{\mathcal{V}}$ over \mathcal{V} , respectively. Then $1 \mapsto t$ defines a log-structure on $\mathcal{C}_{\mathcal{V}}$ and we indicate it by \mathcal{N} . After shrinking C it is also possible to endow $(\mathcal{C}_{\mathcal{V}}, \mathcal{N})$ with a lift $\sigma_{\mathcal{C}_{\mathcal{V}}}$ of Frobenius which is compatible with the Frobenius σ on \mathcal{V} . We then have a sequence of exact closed immersions of log-schemes

$$(5) \quad s^\times \rightarrow (C, s) \rightarrow (\mathcal{C}_{\mathcal{V}}, \mathcal{N}).$$

The log-scheme (C, s) in (5) is log-smooth over k endowed with the trivial log-structure, and the formal log-scheme $(\mathcal{C}_{\mathcal{V}}, \mathcal{N})$ in (5) is log-smooth over \mathcal{V} endowed with the trivial log-structure. We will denote the reduction of $(\mathcal{C}_{\mathcal{V}}, \mathcal{N})$ modulo p^i by $(\mathcal{C}_{\mathcal{V}_i}, \mathcal{N}_i)$. We will indicate by \mathcal{C}_K (resp. $(\mathcal{C}_K, \mathcal{N}_K)$) the rigid analytic space associated to the generic fiber of $\mathcal{C}_{\mathcal{V}}$ (resp. with the log-structure \mathcal{N}_K induced by \mathcal{N}), and denotes by \hat{s}_K a point of \mathcal{C}_K defined by $t = 0$.

As in paragraph 2 we may induce on X_s the log-structure of (X, M) and we refer to it as (X_s, M_s) . Again (X, M) is log-smooth over k endowed with the trivial log-structure while (X_s, M_s) is log-smooth over s^\times . Then we have the following diagram:

$$(6) \quad \begin{array}{ccccc} (X_s, M_s) & \longrightarrow & (X, M) & & \\ \downarrow f_s & & \downarrow f & & \\ s^\times & \longrightarrow & (C, s) & \longrightarrow & (\mathcal{C}_{\mathcal{V}}, \mathcal{N}). \end{array}$$

In this setting Kato and, later, Shiho ([KA89], [SH08], [SH08A]) were able to define the *relative log-crystalline cohomology sheaves of $(X, M)/(C, s)$ with respect to $(\mathcal{C}_{\mathcal{V}}, \mathcal{N})$* , and we will indicate them by

$$\mathbb{R}^m f_{(X, M)/(C, s), \text{crys}^*(\text{crys}, X, K)}.$$

In this paper we will only work with the trivial log-isocrystal $_{\text{crys}, X, K}$. In order to define the relative log-crystalline cohomology one needs to fix a Hyodo-Kato embedding system $(\mathcal{P}_\bullet, \mathcal{M}_\bullet)$ of an étale

hypercovering (X_\bullet, M_\bullet) of (X, M) [HK94] 2.18 (see the notion of simplicial (formal) schemes and étale hypercovering in [CT03], [TS04]):

$$(7) \quad \begin{array}{ccccc} (X_{s,\bullet}, M_{s,\bullet}) & \longrightarrow & (X_\bullet, M_\bullet) & \xrightarrow{i_\bullet} & (\mathcal{P}_\bullet, \mathcal{M}_\bullet) \\ \downarrow \theta_s & & \downarrow \theta & & \downarrow g \\ (X_s, M_s) & \longrightarrow & (X, M) & & \downarrow \\ \downarrow f_s & & \downarrow f & & \downarrow \\ s^\times & \longrightarrow & (C, s) & \longrightarrow & (\mathcal{C}_\mathcal{V}, \mathcal{N}) \end{array}$$

where

- (i) $\theta : X_\bullet \rightarrow X$ is an étale hypercovering, M_\bullet is the log-structure on X_\bullet induced by M , and $\theta_s : (X_{s,\bullet}, M_{s,\bullet}) \rightarrow (X_s, M_s)$ is an induced étale hypercovering by base-change;
- (ii) $(\mathcal{P}_\bullet, \mathcal{M}_\bullet)$ is a simplicial formal log-scheme separated of finite type over \mathcal{V} (with Frobenius endomorphism which extends that of \mathcal{V}) such that $(\mathcal{P}_\bullet, \mathcal{M}_\bullet)$ is a simplicial formal log-scheme over $(\mathcal{C}_\mathcal{V}, \mathcal{N})$ and $(\mathcal{P}_\bullet, \mathcal{M}_\bullet)$ is log-smooth over $(\mathcal{C}_\mathcal{V}, \mathcal{N})$;
- (iii) i_\bullet is a closed immersion of simplicial formal log-schemes (not necessary exact).

Then the *crystalline complex* $C_{(X_\bullet, M_\bullet)/(\mathcal{C}_\mathcal{V}, \mathcal{N})}$ on the Zariski site on X_\bullet can be defined by the logarithmic de Rham complex of the PD envelop of the closed immersion $i_\bullet : (X_\bullet, M_\bullet) \rightarrow (\mathcal{P}_\bullet, \mathcal{M}_\bullet)$ over $(\mathcal{C}_\mathcal{V}, \mathcal{N})$ and the relative log-crystalline cohomology is calculated by

$$\mathbb{R}f_{(X, M)/(\mathcal{C}_\mathcal{V}, \mathcal{N}), \text{crys}*}(\text{crys}, X, K) \cong K \otimes \varprojlim_i \mathbb{R}(f\theta)_* C_{(X_\bullet, M_\bullet)/(\mathcal{C}_\mathcal{V}, \mathcal{N})}.$$

Note that our crystalline complex is on the Zariski site after (1.5) in [SH08], and it is not on the étale site as in [KA89] 6.2, 6.4 and [HK94] 2.19.

Remark 3.1. In paragraph 4 we construct an embedding system such that i_\bullet is an exact closed immersion. In this section we do not need such a special embedding system.

We recall the notion of iso-coherent sheaves. The category of iso-coherent sheaves on the Zariski site of a p -adic formal scheme \mathcal{S} is the category such that the objects are those of the category of coherent sheaves on the Zariski site of \mathcal{S} and the group of homomorphisms is given by $\text{Hom}(\mathcal{F}, \mathcal{G}) = \text{Hom}_{\text{Coh}}(\mathcal{F}, \mathcal{G}) \otimes \mathbb{Q}$, where $\text{Hom}_{\text{Coh}}(\mathcal{F}, \mathcal{G})$ is the abelian group of homomorphisms as coherent sheaves.

Theorem 3.2. *Under these hypotheses on (X, M) , $\mathbb{R}f_{(X, M)/(\mathcal{C}_\mathcal{V}, \mathcal{N}), \text{crys}*}(\text{crys}, X, K)$ is a perfect complex of iso-coherent sheaves on $\mathcal{C}_\mathcal{V}$; in particular, $\mathbb{R}^m f_{(X, M)/(\mathcal{C}_\mathcal{V}, \mathcal{N}), \text{crys}*}(\text{crys}, X, K)$ is zero for $m \gg 0$. Moreover, for each m , the iso-coherent cohomology sheaf $\mathbb{R}^m f_{(X, M)/(\mathcal{C}_\mathcal{V}, \mathcal{N}), \text{crys}*}(\text{crys}, X, K)$ admits a Frobenius structure.*

Proof. See [HK94], 2.24. Note that the map is of Cartier type. \square

Moreover Shiho ([SH08] 1.19) was also able, always in our setting (6), to define a base change theorem (see also [KA89] 6.10). In fact one can complete the diagram (6) as

$$(8) \quad \begin{array}{ccccc} (X_s, M_s) & \xrightarrow{f_s} & s^\times & \longrightarrow & \hat{s}^\times = \mathcal{V}^\times \\ \downarrow \iota & & \downarrow \iota & & \downarrow \hat{\iota} \\ (X, M) & \xrightarrow{f} & (C, s) & \longrightarrow & (\mathcal{C}_\mathcal{V}, \mathcal{N}), \end{array}$$

where we indicate the morphism defined by $t \rightarrow 0$ by $\hat{\iota}$ and identify \hat{s}^\times with \mathcal{V}^\times . Note that all the squares are cartesian. The ι 's and $\hat{\iota}$ are exact closed immersions and f and f_s are proper and log-smooth. Then, following [SH08]1.19:

Theorem 3.3. *As in (8) we have an isomorphism*

$$\mathbb{L}^* \mathbb{R}f_{(X,M)/(\mathcal{C}_{\mathcal{V}}, \mathcal{N}), \text{crys}*}(\text{crys}, X, K) \cong \mathbb{R}f_{s(X_s, M_s)/\mathcal{V}^\times, \text{crys}*}(\text{crys}, X_s, K)$$

in the derived category of perfect K -complexes.

Moreover the relative log-crystalline cohomology groups of (X_s, M_s) over s^\times are nothing but the log-crystalline cohomology of (X_s, M_s) over \mathcal{V}^\times in the sense of [HK94]. Then we may understand the foregoing base-change theorem as an identification of K -vector spaces

$$(9) \quad \mathbb{R}^m f_{(X,M)/(\mathcal{C}_{\mathcal{V}}, \mathcal{N}), \text{crys}*}(\text{crys}, X, K)_{s^\times} \cong H_{\log\text{-crys}}^m((X_s, M_s)/\mathcal{V}^\times) \otimes K.$$

To prove this identification (9) (resp. the similar identifications in (11) below), we will need to use the local projectivity of $\mathbb{R}f_{(X,M)/(\mathcal{C}_{\mathcal{V}}, \mathcal{N}), \text{crys}*}(\text{crys}, X, K)$. This will follow from the locally freeness of relative log-analytic cohomology sheaves (Theorem 3.8), via the isomorphism of Theorem 3.6 and the identification of Theorem 3.4. In order to make evident our approach we decided to write the isomorphism (9) at this point of the article.

* * *

We will need not only relative log-crystalline cohomology, but also the entire apparatus developed by Shiho in his work: namely two other relative cohomology theories: the log-convergent and the log-analytic cohomologies. In our setting, Shiho [SH08], [SH08A] was able to introduce the *relative m -th log-convergent cohomology sheaves of $(X, M)/(C, s)$* with respect to $(\mathcal{C}_{\mathcal{V}}, \mathcal{N})$ which are indicated by

$$\mathbb{R}^m f_{(X,M)/(\mathcal{C}_{\mathcal{V}}, \mathcal{N}), \text{conv}*}(\text{conv}, X, K).$$

Here conv, X, K is the trivial convergent isocrystal. If we denote the log-tube of the closed immersion $i_\bullet : (X_\bullet, M_\bullet) \rightarrow (\mathcal{P}_\bullet, \mathcal{M}_\bullet)$, i.e., the usual tube $]X_\bullet[_{\mathcal{P}_\bullet}^{\text{ex}}$ of the exactification of i_\bullet , by $]X_\bullet[_{\mathcal{P}_\bullet}^{\log}$, and if $\text{sp} :]X_\bullet[_{\mathcal{P}_\bullet}^{\log} \rightarrow X_\bullet$ is the specialization map [SH08] 2.19, then the log-convergent cohomology sheaves are calculated by the logarithmic de Rham complex

$$\Omega_{]X_\bullet[_{\mathcal{P}_\bullet}^{\log}/\mathcal{C}_K}^\bullet < \mathcal{M}_\bullet / \mathcal{N} >$$

of the simplicial rigid analytic space $]X_\bullet[_{\mathcal{P}_\bullet}^{\log}$ over $\mathcal{C}_K =]C[_{\mathcal{C}_{\mathcal{V}}} =]C[_{\mathcal{C}_{\mathcal{V}}}^{\log}$ (because the closed immersion $(C, s) \rightarrow (\mathcal{C}_{\mathcal{V}}, \mathcal{V})$ is exact) [SH08] 2.34:

$$\mathbb{R}f_{(X,M)/(\mathcal{C}_{\mathcal{V}}, \mathcal{N}), \text{conv}*}(\text{conv}, X, K) \cong \mathbb{R}(f\theta)_* \text{sp}_* \Omega_{]X_\bullet[_{\mathcal{P}_\bullet}^{\log}/\mathcal{C}_K}^\bullet < \mathcal{M}_\bullet / \mathcal{N} > .$$

Then there is a canonical comparison morphism

$$(10) \quad \text{sp}_* \Omega_{]X_\bullet[_{\mathcal{P}_\bullet}^{\log}/\mathcal{C}_K}^\bullet < \mathcal{M}_\bullet / \mathcal{N} > \longrightarrow K \otimes \varprojlim_i C_{(X_\bullet, M_\bullet)/(\mathcal{C}_{\mathcal{V}_i}, \mathcal{N}_i)}$$

and it induces the comparison theorem in [SH08] 2.36:

Theorem 3.4. *The canonical morphism (10) induces an isomorphism*

$$\mathbb{R}^m f_{(X,M)/(\mathcal{C}_{\mathcal{V}}, \mathcal{N}), \text{conv}*}(\text{conv}, X, K) \cong \mathbb{R}^m f_{(X,M)/(\mathcal{C}_{\mathcal{V}}, \mathcal{N}), \text{crys}*}(\text{crys}, X, K)$$

of iso-coherent sheaves on $\mathcal{C}_{\mathcal{V}}$ such that the Frobenius structures on both sides commute.

Remark 3.5. To any log-convergent isocrystal, \mathcal{E} , it is possible to associate a log-crystalline isocrystal $\Phi(\mathcal{E})$ [SH08] 2.35. Here we should have written $\Phi(\mathcal{O}_{\text{conv}, X, K})$ for the structural log-convergent isocrystal $\mathcal{O}_{\text{crys}, X, K}$. But $\Phi(\mathcal{O}_{\text{conv}, X, K})$ is the structural log-crystalline isocrystal. Hence we prefer to omit Φ .

As a corollary of Theorem 3.4, we have that the relative log-convergent cohomology sheaves of $(X, M)/(\mathcal{C}_V, \mathcal{N})$ are endowed with a Frobenius structure and we have, as before, a base change theorem for relative log-convergent cohomology (as K -vector spaces) $H_{\log\text{-conv}}^m((X_s, M_s)/\mathcal{V}^\times)$ is isomorphic to $\mathbb{R}^m f_{(X,M)/(\mathcal{C}_V, \mathcal{N}), \text{conv}*}(X, K)_{s^\times}$. We summarize all of these with a diagram where all the maps are isomorphisms (for all m) of K -vector spaces:

$$(11) \quad \begin{array}{ccc} H_{\log\text{-conv}}^m((X_s, M_s)/\mathcal{V}^\times) & \longrightarrow & H_{\log\text{-crys}}^m((X_s, M_s)/\mathcal{V}^\times) \otimes K \\ \downarrow & & \downarrow \\ \mathbb{R}^m f_{(X,M)/(\mathcal{C}_V, \mathcal{N}), \text{conv}*}(\text{conv}, X, K)_{s^\times} & \longrightarrow & \mathbb{R}^m f_{(X,M)/(\mathcal{C}_V, \mathcal{N}), \text{crys}*}(\text{crys}, X, K)_{s^\times}. \end{array}$$

* * *

With regard to the results we have obtained thus far, we know only that the iso-coherent sheaves $\mathbb{R}^m f_{(X,M)/(\mathcal{C}_V, \mathcal{N}), \text{conv}*}(\text{conv}, X, K)$ on \mathcal{C}_V are endowed with a Frobenius structure, but we have not given any information at the differential level. Thus we must introduce, after [SH08], the *relative log-analytic cohomology sheaves of $(X, M)/(C, s)$* with respect to $(\mathcal{C}_V, \mathcal{N})$ which is indicated and defined by

$$\mathbb{R}^m f_{(X,M)/(\mathcal{C}_V, \mathcal{N}), \text{an}*}(\text{an}, X, K) = \mathbb{R}^m g_{K*}^{\text{ex}} \Omega_{[X_\bullet]_{\mathcal{P}_\bullet}^{\log}/\mathcal{C}_K}^\bullet < \mathcal{M}_\bullet / \mathcal{N} >.$$

in [SH08] 4.1, where $g_K^{\text{ex}} : [X_\bullet]_{\mathcal{P}_\bullet}^{\log} \rightarrow \mathcal{C}_K$ is the induced morphism from $g : \mathcal{P}_\bullet \rightarrow \mathcal{C}_V$. Then they are \mathcal{C}_K -sheaves. By [SH08] 4.6 we have

Theorem 3.6. *In the previous notation, the relative log-analytic cohomology $\mathbb{R}^m f_{(X,M)/(\mathcal{C}_V, \mathcal{N}), \text{an}*}(\text{an}, X, K)$ are coherent \mathcal{C}_K -sheaves. Moreover we have an isomorphism for any m :*

$$\text{sp}_* \mathbb{R}^m f_{(X,M)/(\mathcal{C}_V, \mathcal{N}), \text{an}*}(\text{an}, X, K) \cong \mathbb{R}^m f_{(X,M)/(\mathcal{C}_V, \mathcal{N}), \text{conv}*}(\text{conv}, X, K),$$

compatible with Frobenius maps ($\text{sp} : \mathcal{C}_K \rightarrow \mathcal{C}_V$ is the specialization morphism).

We should stress the fact that those objects do not have the structures of isocrystals, i.e., they may be not isocrystals on the log-convergent site $((C, s)/\mathcal{V})_{\text{conv}}^{\log}$ where \mathcal{V} is endowed with the trivial log-structure. But this is almost the case. In fact Shiho proved [SH08] 4.8, 4.10:

Theorem 3.7. *In the previous notation, for any integer m , there exists a unique coherent convergent isocrystal \mathcal{F} on $((C, s)/\mathcal{V})_{\text{conv}}^{\log}$ such that for any pre-widening $(\mathcal{Z}, \mathcal{N}_{\mathcal{Z}}, Z, M_Z, i, z)$ such that $z : Z \rightarrow C$ is a strict morphism and $(Z, \mathcal{N}_{\mathcal{Z}})$ is formally log-smooth over \mathcal{V} endowed with the trivial log-structure, the restriction of \mathcal{F} to $I_{\text{conv}}(Z/\mathcal{V})^{\log} = \text{Str}'(Z \rightarrow \mathcal{Z}/\mathcal{V})^{\log}$ is given in a functorial way by*

$$\mathbb{R}^m f_{(X,M) \times_{(C,s)} (\mathcal{Z}, \mathcal{N}_{\mathcal{Z}})/(\mathcal{Z}, \mathcal{N}_{\mathcal{Z}}), \text{conv}*}(\text{conv}, X, K)$$

endowed with a stratification. An analogous statement holds for the log-analytic setting where this time we will speak of a structure of log-module with connection.

In particular, our lifting $(\mathcal{C}_V, \mathcal{N})$ fits in the hypotheses of the previous theorem and we conclude that $\mathbb{R}^m f_{(X,M)/(\mathcal{C}_V, \mathcal{N}), \text{an}*}(\text{an}, X, K)$ is a coherent sheaf endowed with a log-connection on $(\mathcal{C}_K, \mathcal{N}_K)$. This connection is that of Gauss-Manin for

$$(X, M) \rightarrow (C, s) \rightarrow k$$

(k endowed with the trivial log-structure) as explained in Shiho's Theorems 4.8 and 4.10 of [SH08]. Moreover the isomorphism of Theorem 3.6 gives to $\mathbb{R}^m f_{(X,M)/(\mathcal{C}_V, \mathcal{N}), \text{an}*}(\text{an}, X, K)$ also a Frobenius structure as an iso-coherent sheaf on \mathcal{C}_K . In fact $\mathbb{R}^m f_{(X,M)/(\mathcal{C}_V, \mathcal{N}), \text{an}*}(\text{an}, X, K)$ can be endowed with a Frobenius map, which becomes an isomorphism after applying the specialization morphism $\text{sp} : \mathcal{C}_K \rightarrow \mathcal{C}_V$ by the comparison theorems 3.6, 3.4 and the log-crystalline result 3.2. However, if $\text{sp}_* \mathcal{F} = 0$ for a coherent sheaf \mathcal{F} on \mathcal{C}_K , then one can argue that $\mathcal{F} = 0$, and so we may conclude that the Frobenius map is an isomorphism for $\mathbb{R}^m f_{(X,M)/(\mathcal{C}_V, \mathcal{N}), \text{an}*}(\text{an}, X, K)$. We may then state:

Theorem 3.8. *In the previous notation, the relative log-analytic cohomology $\mathbb{R}^m f_{(X,M)/(\mathcal{C}_V, \mathcal{N}), \text{an}^*(\text{an}, X, K)}$ is a locally free sheaf on \mathcal{C}_K endowed with a Frobenius structure and a logarithmic connection on $(\mathcal{C}_K, \mathcal{N}_K)$.*

Proof. Outside \hat{s}_K in \mathcal{C}_K , the log-connection is a usual connection, hence the sheaf is locally free, because we are in characteristic 0. The problem is at \hat{s}_K . Here it is enough to check that there is no nontrivial torsion. The existence of the Frobenius structure forces an isomorphism between the original module and its transform by Frobenius. If we had a nontrivial torsion we would have an isomorphism between modules with different lengths. This is a contradiction. \square

As a summary of the results collected in this paragraph, so far, we have the following isomorphism for the residue at \hat{s}_K^\times of $\mathbb{R}^m f_{(X,M)/(\mathcal{C}_V, \mathcal{N}), \text{an}^*(\text{an}, X, K)}$ (i.e. as in (8), $t \mapsto 0$) in \mathcal{C}_K :

$$(12) \quad \mathbb{R}^m f_{(X,M)/(\mathcal{C}_V, \mathcal{N}), \text{an}^*(\text{an}, X, K)}|_{\hat{s}_K^\times} \cong H_{\log\text{-crys}}^m((X_s, M_s)/\mathcal{V}^\times) \otimes K.$$

This isomorphism is compatible with Frobenius structures on both sides.

* * *

Since $\mathbb{R}^m f_{(X,M)/(\mathcal{C}_V, \mathcal{N}), \text{an}^*(\text{an}, X, K)}$ is locally free on \mathcal{C}_K , it is free on $]s[_{\mathcal{C}_V} = D(0, 1^-)$ (the unit open disc) [CR98] 4.9 and there exists a K -vector space V_m endowed with a nilpotent endomorphism N_m and a Frobenius structure F_m satisfying $N_m F_m = p F_m N_m$ such that

$$(13) \quad (\mathbb{R}^m f_{(X,M)/(\mathcal{C}_V, \mathcal{N}), \text{an}^*(\text{an}, X, K)}|_{]s[_{\mathcal{C}_V}}, \nabla, \varphi_m) \cong (V_m \otimes]s[_{\mathcal{C}_V}, N_m \otimes 1 + 1 \otimes d, F_m \otimes \sigma_{\mathcal{C}_V})$$

and V_m is isomorphic to $H_{\log\text{-crys}}^m((X_s, M_s)/\mathcal{V}^\times) \otimes K$ by (12). Here ∇ (resp. φ_m) is a connection (resp. a Frobenius structure) on $\mathbb{R}^m f_{(X,M)/(\mathcal{C}_V, \mathcal{N}), \text{an}^*(\text{an}, X, K)}$. Indeed, all solutions of the associated differential equations are convergent on the unit disc $]s[_{\mathcal{C}_V}$ by the Frobenius structure. (V_m, N_m, F_m) is called the residue of $\mathbb{R}^m f_{(X,M)/(\mathcal{C}_V, \mathcal{N}), \text{an}^*(\text{an}, X, K)}$ at \hat{s}_K^\times .

Now on the right part of (12) we have a monodromy operator from [HK94] 3.6 while on the left hand side we have the residue endomorphism N_m . We would like now to prove the following

Theorem 3.9. *In the isomorphism (12) the two aforementioned monodromy operators are the same.*

Proof. We use the construction of the monodromy operator of $H_{\log\text{-crys}}^m((X_s, M_s)/\mathcal{V}^\times) \otimes K$ as in [HK94] 3.6. Shrinking C , we may assume that $\mathcal{C}_V \rightarrow \text{Spf } \widehat{\mathcal{V}[t]}$ is étale where t is a local coordinate of \hat{s} over \mathcal{V} and $\mathcal{V}[t]$ is a p -adic completion of the polynomial ring $\mathcal{V}[t]$.

Let $C_{(X,M)/\mathcal{V}_i}$ the crystalline complex of (X, M) over \mathcal{V}_i endowed with the trivial log-structure on the Zariski site on X_\bullet . We then have an exact sequence for each n as in [HK94] 3.6:

$$(14) \quad 0 \rightarrow C_{(X,M)/(\mathcal{C}_{\mathcal{V}_i}, \mathcal{N}_i)}[-1] \rightarrow C_{(X,M)/\mathcal{V}_i} \rightarrow C_{(X,M)/(\mathcal{C}_{\mathcal{V}_i}, \mathcal{N}_i)} \rightarrow 0,$$

where the first map is given by the external multiplication by $\omega \mapsto dt/t \wedge \omega$. Note that $(C^\bullet, d^\bullet)[-1] = (C^{\bullet-1}, -d^{\bullet-1})$ by definition. The exact sequence

$$(15) \quad 0 \rightarrow C_{(X,M)/(\mathcal{C}_{\mathcal{V}_i}, \mathcal{N}_i)} \otimes_{\mathcal{O}_{\mathcal{C}_{\mathcal{V}_i}}} \mathcal{V}_i[-1] \rightarrow C_{(X,M)/\mathcal{V}_i} \otimes_{\mathcal{O}_{\mathcal{C}_{\mathcal{V}_i}}} \mathcal{V}_i \rightarrow C_{(X,M)/(\mathcal{C}_{\mathcal{V}_i}, \mathcal{N}_i)} \otimes_{\mathcal{O}_{\mathcal{C}_{\mathcal{V}_i}}} \mathcal{V}_i \rightarrow 0$$

induced by tensoring (14) with $\mathcal{O}_{\mathcal{C}_{\mathcal{V}_i}} \rightarrow \mathcal{V}_i$ ($t \mapsto 0$) is nothing but the second exact sequence in [HK94] 3.6 since $\mathcal{C}_V \rightarrow \text{Spf } \widehat{\mathcal{V}[t]}$ is étale. Hence the connecting homomorphisms on the cohomology groups with respect to the short exact sequence (15) induce the monodromy operators on $H_{\log\text{-crys}}^m((X_s, M_s)/\mathcal{V}^\times) \otimes K$ by the isomorphism

$$H_{\log\text{-crys}}^m((X_s, M_s)/\mathcal{V}^\times) \otimes K \cong \varprojlim_{\mathfrak{s}^\times} \mathbb{R}^m(f_s \theta_s)_* C_{(X,M)/(\mathcal{C}_{\mathcal{V}_i}, \mathcal{N}_i, \mathfrak{s}^\times)} \otimes K,$$

where $C_{(X,M)/(\mathcal{C}_{\mathcal{V}_i}, \mathcal{N}_i, \mathfrak{s}^\times)}$ is the fiber of $C_{(X,M)/(\mathcal{C}_{\mathcal{V}_i}, \mathcal{N}_i)}$ at \mathfrak{s}^\times and it is isomorphic to the crystalline complex of $(X_s, M_s)/\mathfrak{s}^\times$ with respect to the embedding system $(X_{s,\bullet}, M_{s,\bullet}) \rightarrow (\mathcal{P}_\bullet, \mathcal{M}_\bullet) \otimes_{\mathcal{C}_V} \mathcal{V}$ ($\mathcal{C}_V \rightarrow \mathcal{V}$, $t \mapsto 0$) induced from the diagram (7).

A similar procedure holds for the log-convergent cohomology: there is an exact sequence

$$(16) \quad 0 \rightarrow \Omega_{|X_\bullet|_{\mathcal{P}_\bullet}^{\log}/\mathcal{C}_K}^\bullet < \mathcal{M}_\bullet/\mathcal{N} > [-1] \rightarrow \Omega_{|X_\bullet|_{\mathcal{P}_\bullet}^{\log}}^\bullet < \mathcal{M}_\bullet > \rightarrow \Omega_{|X_\bullet|_{\mathcal{P}_\bullet}^{\log}/\mathcal{C}_K}^\bullet < \mathcal{M}_\bullet/\mathcal{N} > \rightarrow 0$$

where $\Omega_{|X_\bullet|_{\mathcal{P}_\bullet}^{\log}}^\bullet < \mathcal{M}_\bullet >$ is the logarithmic de Rham complex of $(X, M)/K$ with respect to the embedding system (7) and the first map is given by the external multiplication by $\omega \mapsto dt/t \wedge \omega$. Consider the induced short exact sequence by tensoring with $\mathcal{O}_{\mathcal{C}_K} \rightarrow K (t \mapsto 0)$ and taking cohomology sheaves with respect to $\mathbb{R}(f_s\theta_s)_*\mathrm{sp}_*$, the connecting homomorphism is the monodromy operator on $H_{\log\text{-conv}}^m((X_s, M_s)/\mathcal{V}^\times)$ which is compatible with the monodromy operator on the crystalline cohomology by the comparison isomorphism of the diagram (11).

The connection on the relative log-convergent cohomology $\mathbb{R}^m f_{(X,M)/(\mathcal{C}_V, \mathcal{N}), \text{conv}*}(\text{conv}, X, K)$, hence on the relative log-analytic cohomology $\mathbb{R}^m f_{(X,M)/(\mathcal{C}_V, \mathcal{N}), \text{an}*}(\text{an}, X, K)$, is the Gauss-Manin connection for the composite of log-smooth morphisms

$$(X, M) \rightarrow (C, s) \rightarrow k,$$

(where k is endowed with the trivial log-structure). It follows that the connection is given by the edge homomorphism of E_1 -terms of the Leray spectral sequences of the composite of morphisms (see [KO68]. sect. 4). Hence, the connection on the relative log-convergent (log-analytic) cohomology is given by the connecting homomorphism arising from the short exact sequence (16). Therefore the two monodromy operators on $H_{\log\text{-crys}}^m((X_s, M_s)/\mathcal{V}^\times) \otimes K$ and V_m under the isomorphism (12) coincide with each other. \square

Corollary 3.10. *The isomorphism (12) induces an isomorphism*

$$V_m \cong H_{\log\text{-crys}}^m((X_s, M_s)/\mathcal{V}^\times) \otimes K$$

of K -spaces which is compatible with nilpotent endomorphisms and Frobenius structures.

Hence, by applying $\mathbb{R}\Gamma(|X_{s,\bullet}|_{\mathcal{P}_\bullet}, -)$ to (16), we obtain an exact sequence of K -vector spaces

$$\begin{aligned} \cdots \rightarrow H_{\log\text{-conv}}^m((X_s, M_s)/\mathcal{V}) &\rightarrow \Gamma(|s|_{\mathcal{C}_V}, \mathbb{R}^m f_{(X,M)/(\mathcal{C}_V, \mathcal{N}), \text{an}*}(\text{an}, X, K)) \\ &\xrightarrow{\nabla} \Gamma(|s|_{\mathcal{C}_V}, \mathbb{R}^m f_{(X,M)/(\mathcal{C}_V, \mathcal{N}), \text{an}*}(\text{an}, X, K)) dt/t \rightarrow \cdots \end{aligned}$$

where ∇ is the log-connection on the free $\Gamma(|s|_{\mathcal{C}_V}, |s|_{\mathcal{C}_V})$ -module endowed with a Frobenius structure. Taking the residue of $\Gamma(|s|_{\mathcal{C}_V}, \mathbb{R}^m f_{(X,M)/(\mathcal{C}_V, \mathcal{N}), \text{an}*}(\text{an}, X, K))$ at \hat{s}_K^\times (see (13)), we have the following long exact sequence of finite dimensional K -vector spaces

$$(3) \quad \cdots \rightarrow H_{\log\text{-conv}}^m((X_s, M_s)/\mathcal{V}) \rightarrow H_{\log\text{-crys}}^m((X_s, M_s)/\mathcal{V}^\times) \otimes K \xrightarrow{N_m} H_{\log\text{-crys}}^m((X_s, M_s)/\mathcal{V}^\times) \otimes K(-1) \rightarrow \cdots$$

by Corollary 3.10.

4 Building up the sequence

In order to calculate the rigid cohomology of X (for example) one does not really need to take a covering of the variety respecting the fact that it is defined over C . But for future use we will need the existence of coverings which are compatible with the map to the curve C and, possibly, with a smooth compactification. So, we are always in the diagram (6). Let \overline{C} denote a compactification of C . By Nagata compactification of X over \overline{C} we may think of having a compactification \overline{X} of X over \overline{C} . We may then suppose that we have a simplicial Zariski hypercovering diagram of the type

$$\begin{array}{ccccc} X_{s,\bullet} & \xrightarrow{\iota_\bullet} & X_\bullet & \longrightarrow & \overline{X}_\bullet \longrightarrow \mathcal{P}_\bullet \\ \downarrow & & \downarrow & & \downarrow \\ X_s & \longrightarrow & X & \longrightarrow & \overline{X} \\ \downarrow & & \downarrow & & \downarrow \\ s & \longrightarrow & C & \longrightarrow & \overline{C}. \end{array}$$

The simplicial map $\overline{X}_\bullet \rightarrow \overline{X}$ is a Zariski affine hypercovering, \mathcal{P}_\bullet is a simplicial formal scheme separated of finite type over \mathcal{V} which is smooth around X_\bullet and admits a lift σ_\bullet of Frobenius compatible with the Frobenius σ on \mathcal{V} and all squares are cartesian. Such a setting allows us to calculate $H_{X_s, \text{rig}}^m(X)$. In fact we have:

$$(17) \quad H_{X_s, \text{rig}}^m(X) = \mathbb{R}^m \Gamma(\overline{X}_\bullet[\mathcal{P}_\bullet, [j_{X_\bullet[\mathcal{P}_\bullet]}^\dagger \Omega_{\overline{X}_\bullet[\mathcal{P}_\bullet]}^\bullet \rightarrow j_{X_\bullet \setminus X_s, \bullet[\mathcal{P}_\bullet]}^\dagger \Omega_{\overline{X}_\bullet[\mathcal{P}_\bullet]}^\bullet]],$$

where $j_{U_\bullet[\mathcal{P}_\bullet]}^\dagger \mathcal{F}_\bullet$ is a simplicial sheaves of overconvergent sections along $\overline{X}_\bullet \setminus U_\bullet[\mathcal{P}_\bullet]$ associated to a sheaf \mathcal{F}_\bullet of abelian groups on $\overline{X}_\bullet[\mathcal{P}_\bullet]$ for a simplicial open subscheme U_\bullet of \overline{X}_\bullet . Here j is seen as the map: $j_{U_\bullet[\mathcal{P}_\bullet]} : U_\bullet[\mathcal{P}_\bullet] \rightarrow \overline{X}_\bullet[\mathcal{P}_\bullet]$. In particular we will have a long exact sequence

$$\cdots \rightarrow H_{X_s, \text{rig}}^m(X) \rightarrow H_{\text{rig}}^m(X) \rightarrow H_{\text{rig}}^m(X \setminus X_s) \rightarrow \cdots$$

of finite dimensional K -vector spaces with Frobenius structures.

In order to construct the exact sequence we seek, we will need to deal with complexes which are defined over $\overline{X}_\bullet[\mathcal{P}_\bullet]$. We may then consider an admissible covering of $\overline{X}_\bullet[\mathcal{P}_\bullet]$ given by

$$\{\overline{X}_\bullet[\mathcal{P}_\bullet] \setminus X_{s, \bullet}[\mathcal{P}_\bullet], X_\bullet[\mathcal{P}_\bullet]\}.$$

The two complexes which form the simple complex in (17) are equal on $\overline{X}_\bullet[\mathcal{P}_\bullet] \setminus X_{s, \bullet}[\mathcal{P}_\bullet]$ and in the intersection $(\overline{X}_\bullet[\mathcal{P}_\bullet] \setminus X_{s, \bullet}[\mathcal{P}_\bullet]) \cap X_\bullet[\mathcal{P}_\bullet]$ (note that $X_s \cap (\overline{X} \setminus X) = \emptyset$). We then conclude that

$$(18) \quad [j_{X_\bullet[\mathcal{P}_\bullet]}^\dagger \Omega_{\overline{X}_\bullet[\mathcal{P}_\bullet]}^\bullet \rightarrow j_{X_\bullet \setminus X_{s, \bullet}[\mathcal{P}_\bullet]}^\dagger \Omega_{\overline{X}_\bullet[\mathcal{P}_\bullet]}^\bullet] \cong [\alpha_{\bullet, *} (j_{X_\bullet[\mathcal{P}_\bullet]}^\dagger \Omega_{\overline{X}_\bullet[\mathcal{P}_\bullet]}^\bullet)_{|X_\bullet[\mathcal{P}_\bullet]} \rightarrow \alpha_{\bullet, *} (j_{X_\bullet \setminus X_{s, \bullet}[\mathcal{P}_\bullet]}^\dagger \Omega_{\overline{X}_\bullet[\mathcal{P}_\bullet]}^\bullet)_{|X_\bullet[\mathcal{P}_\bullet]}]$$

where $\alpha_\bullet : X_\bullet[\mathcal{P}_\bullet] \rightarrow \overline{X}_\bullet[\mathcal{P}_\bullet]$ and of course we may re-write the second part of (18) as

$$(19) \quad \alpha_{\bullet, *} [\Omega_{\overline{X}_\bullet[\mathcal{P}_\bullet]}^\bullet \rightarrow j_{X_\bullet \setminus X_{s, \bullet}[\mathcal{P}_\bullet]}^\dagger \Omega_{\overline{X}_\bullet[\mathcal{P}_\bullet]}^\bullet]$$

where now we consider $j_{U_\bullet[\mathcal{P}_\bullet]} : U_\bullet[\mathcal{P}_\bullet] \rightarrow X_\bullet[\mathcal{P}_\bullet]$.

Hence we would like to compare $H_{X_s, \text{rig}}^m(X)$ with

$$(20) \quad H_{X_s, \text{rig}}^m((X, X)) := \mathbb{R}^m \Gamma(X_\bullet[\mathcal{P}_\bullet], [\Omega_{X_\bullet[\mathcal{P}_\bullet]}^\bullet \rightarrow j_{X_\bullet \setminus X_{s, \bullet}[\mathcal{P}_\bullet]}^\dagger \Omega_{X_\bullet[\mathcal{P}_\bullet]}^\bullet])$$

We will refer to such a cohomology $H_{X_s, \text{rig}}^m((X, X))$ as *rigid cohomology of the pair* (X, X) with support in X_s along the terminology of [CT03] and [TS04]. Note that the usual rigid cohomology $H_{X_s, \text{rig}}^m(X)$ of X with support in X_s is the rigid cohomology $H_{X_s, \text{rig}}^m((X, \overline{X}))$ of the pair (X, \overline{X}) with support in X_s . Then we claim:

Proposition 4.1. *There exists an isomorphism*

$$H_{X_s, \text{rig}}^m(X) \cong H_{X_s, \text{rig}}^m((X, X))$$

which is compatible with Frobenius structures.

Proof. We need to prove the acyclicity of α_\bullet on the sheaves which appear in (19). This follows from the fact that the open immersions $X_\bullet[\mathcal{P}_\bullet] \rightarrow \overline{X}_\bullet[\mathcal{P}_\bullet]$ and $X_\bullet \setminus X_{s, \bullet}[\mathcal{P}_\bullet] \rightarrow \overline{X}_\bullet[\mathcal{P}_\bullet]$ are affinoid morphisms. This comes, by base change, from the fact that an open immersion of curves is affine and X_s is a divisor. \square

We would like now to understand the two simplicial complexes which appear in the definition of the rigid cohomology of (X, X) with support on X_s , $H_{X_s, \text{rig}}^m((X, X)) \cong H_{X_s, \text{rig}}^m((X, X))$ (as in (20)), in order to involve it in another long exact sequence. As in [CT03], the first complex gives the rigid cohomology $H_{\text{rig}}^m((X, X))$ of the pair (X, X) , and the second complex gives the rigid cohomology $H_{\text{rig}}^m((X \setminus X_s, X))$ of the pair $(X \setminus X_s, X)$.

Remark 4.2. These cohomologies (without support) have been named "naive" by Berthelot, "convergent" by [LS07] 7.4.4.

What we now discuss is an interpretation of such a complex in terms of log-convergent cohomology. In order to achieve this goal we have to choose a different good embedding than the original \mathcal{P}_\bullet we have used up to now: it will enjoy some exactness properties.

Proposition 4.3. *It is possible to construct a simplicial étale hypercovering X_\bullet of X which admits a closed immersion in a simplicial smooth formal scheme Q_\bullet^{ex} separated of finite type over \mathcal{V} . Moreover if we endow X with the log-structure coming from the NCD X_s , M , and X_\bullet with the induced log-structure M_\bullet , then we can give to Q_\bullet^{ex} a fine and saturated log-structure \mathcal{M}_\bullet in such a way that we have a diagram*

$$\begin{array}{ccc} (X_\bullet, M_\bullet) & \xrightarrow{i_\bullet^{\text{ex}}} & (Q_\bullet^{\text{ex}}, \mathcal{M}_\bullet) \\ \downarrow & & \downarrow \\ (X, M) & \longrightarrow & \mathcal{V} \end{array}$$

where i_\bullet^{ex} is an exact closed immersion of log-schemes and $(Q_\bullet^{\text{ex}}, \mathcal{M}_\bullet)$ is formal log-smooth over \mathcal{V} endowed with the trivial log structure and it admits a lift σ_\bullet of Frobenius which is compatible with the Frobenius σ of \mathcal{V} .

For our strategy of proof we will construct an étale hypercovering X_\bullet such that, for each m , there is a disjoint decomposition $X_m = \coprod_{\alpha \in I_m} X_{m,\alpha}$ (each component is not necessarily connected) which is compatible with the simplicial structure and that the induced log-structure $M_{m,\alpha}$ on $X_{m,\alpha}$ from M has a chart. Then one can explicitly write down the exactness procedure, which is explained in [KA89] 4.10 and [SH02] 2.2.1. In order to construct an étale hypercovering of X , we use the coskeleton functor and, to construct an embedding system, we use a Γ -construction, which were studied in [CT03] 11.2 and [TS04] 7.2, 7.3. The proof will end at Lemma 4.9.

First we recall truncated simplicial (formal) schemes and the coskeleton functors. Let Δ be the standard simplicial category, put $[l] = \{0, 1, \dots, l\}$ to be an object for a nonnegative integer l , and denote the full subcategory of Δ whose set of objects consists of all $[l]$ with $l \leq q$ by $\Delta[q]$ for a nonnegative integer q . A q -truncated simplicial (formal) scheme over a (formal) scheme S is a contravariant functor from $\Delta[q]$ to the category of (formal) schemes over S . For example, a 1-truncated simplicial (formal) scheme $Y_{\bullet \leq 1}$ over S is represented by

$$(21) \quad Y_{\bullet \leq 1} = \left[\begin{array}{ccc} & \xrightarrow{\pi_0} & \\ Y_1 & \xleftarrow{\delta} & Y_0 \\ & \xrightarrow{\pi_1} & \end{array} \right],$$

where $\pi_l : Y_l \rightarrow Y_0$ (resp. $\delta : Y_0 \rightarrow Y_1$) is a morphism over S corresponding to the map $[0] \rightarrow [1]$ ($0 \mapsto l$) (resp. the unique map $[1] \rightarrow [0]$). Then $\pi_0 \circ \delta = \pi_1 \circ \delta = \text{id}_{Y_0}$ hold.

The q -skeleton functor

$$\text{sk}_q^S : (\text{simplicial (formal) schemes over } S) \longrightarrow (q\text{-truncated (formal) simplicial schemes over } S)$$

has a right adjoint cosk_q^S which is called the q -coskeleton functor. For a nonnegative integer m , let $\Delta[q]/[m]$ be a category such that an object is a morphism $\xi : [l_\xi] \rightarrow [m]$ of Δ with $l_\xi \leq q$ and a morphism $\theta : \xi \rightarrow \eta$ is a morphism of $\Delta[q]$ with $\xi = \eta \circ \theta$. Then q -truncated simplicial (formal) scheme $Y_{\bullet \leq q}$ over S induces a contravariant functor from $\Delta[q]/[m]$ to the category of (formal) S -schemes, $\xi \mapsto Y_{l_\xi}$. Then the m -th stage of $\text{cosk}_q^S(Y_{\bullet \leq q})$ is given by

$$(22) \quad \text{cosk}_q^S(Y_{\bullet \leq q})_m = \varprojlim_{\xi \in \text{Ob}(\Delta[q]/[m])} Y_{l_\xi}$$

where the inverse limit is taken over the diagram of S -schemes over $\Delta[q]/[m]$, and the morphism $\pi_\zeta : \text{cosk}_q^S(Y_{\bullet \leq q})_m \rightarrow \text{cosk}_q^S(Y_{\bullet \leq q})_l$ corresponding to a morphism $\zeta : [l] \rightarrow [m]$ is given by the transformation

$$\Delta[q]/[l] \rightarrow \Delta[q]/[m] \quad \xi \mapsto \zeta \circ \xi$$

of categories. More explicitly, let us put

$$\begin{aligned} T_m &= \prod_{\xi \in \text{Ob}(\Delta[q]/[m])} Y_{l_\xi}, \\ S_m &= \prod_{\theta: \xi \rightarrow \eta \in \text{Mor}(\Delta[q]/[m])} Y_{l_\xi} \end{aligned}$$

where the products are taken over S , and define morphisms $h_i : T_m \rightarrow S_m$ over S for $i = 1, 2$ by

$$\begin{aligned} h_1((x_\xi)) &= (y_{\theta: \xi \rightarrow \eta}) \quad y_{\theta: \xi \rightarrow \eta} = (x_\xi), \\ h_2((x_\xi)) &= (y_{\theta: \xi \rightarrow \eta}) \quad y_{\theta: \xi \rightarrow \eta} = \pi_\theta(x_\eta) \end{aligned}$$

where $\pi_\theta : Y_{l_\eta} \rightarrow Y_{l_\xi}$ is the corresponding morphism of (formal) schemes to $\theta : \xi \rightarrow \eta$. Then $\text{cosk}_q^S(Y_{\bullet \leq q})_m$ is isomorphic to the fiber product of the diagram

$$(23) \quad \begin{array}{ccc} \text{cosk}_q^S(Y_{\bullet \leq q})_m & \xrightarrow{h'_1} & T_m \\ h'_2 \downarrow & & \downarrow h_2 \\ T_m & \xrightarrow{h_1} & S_m. \end{array}$$

When $m \leq q$, $\text{cosk}_q^S(Y_{\bullet \leq q})_m$ is isomorphic to Y_m by the composite of h'_1 and the projection $T_m \rightarrow Y_{\text{id}_{[m]}}$ since $\xi = \text{id}_{[m]} \circ \xi$ for any object ξ of $\Delta[q]/[m]$. If $Y_{\bullet \leq q}$ is separated over S , then $\text{cosk}_q^S(Y_{\bullet \leq q})_m$ is a closed (formal) subscheme of T_m by h'_1 since h_1 is a closed immersion. Moreover, if $Y_{\bullet \leq q}$ is a q -truncated étale hypercovering of S , then $Y_{\bullet \leq q}$ is closed and open in T_m .

Let us consider $\text{cosk}_1^S(Y_{\bullet \leq 1})$ for a 1-truncated simplicial (formal) scheme as in (21). The set of objects of $\Delta[1]/[m]$ consists of

$$\begin{aligned} \xi_i : [0] &\rightarrow [m] & \xi_i(0) &= i \quad (0 \leq i \leq m), \\ \eta_{i,j} : [1] &\rightarrow [m] & \eta_{i,j}(0) &= i, \eta_{i,j}(1) = j \quad (0 \leq i < j \leq m), \\ \zeta_i : [1] &\rightarrow [m] & \zeta_i(0) &= \zeta_i(1) = i \quad (0 \leq i \leq m). \end{aligned}$$

Then $\text{cosk}_1^S(Y_{\bullet \leq 1})_m$ is characterized in T_m via the closed immersion h'_1 in (23) by the following lemma:

Lemma 4.4. $x \in T_m$ belongs to $\text{cosk}_1^S(Y_{\bullet \leq 1})_m$ if and only if x simultaneously satisfies the conditions:

$$\begin{aligned} p_{\xi_i}(x) &= \pi_0 \circ p_{\eta_{i,j}}(x) & \text{for } 0 \leq i < j \leq m, \\ p_{\xi_j}(x) &= \pi_1 \circ p_{\eta_{i,j}}(x) & \text{for } 0 \leq i < j \leq m, \\ p_{\zeta_i}(x) &= \delta \circ p_{\xi_i}(x) & \text{for } 0 \leq i \leq m, \end{aligned}$$

where $p_{\xi_i} : T_m \rightarrow Y_{l_{\xi_i}}$ (resp. $p_{\eta_{i,j}} : T_m \rightarrow Y_{l_{\eta_{i,j}}}$, resp. $p_{\zeta_i} : T_m \rightarrow Y_{l_{\zeta_i}}$) denotes the projection.

Proof. The conditions in the assertion are the relations $\pi_{\xi_i} = \pi_0 \circ \pi_{\eta_{i,j}}, \pi_{\xi_j} = \pi_1 \circ \pi_{\eta_{i,j}}, \pi_{\zeta_i} = \delta \circ \pi_{\xi_i}$ which comes from morphisms of $\Delta[1]/[m]$, respectively. All other relations deduce from these relations. \square

>From now we begin a proof of Proposition 4.3. Let us take a 1-truncated simplicial scheme $X_{\bullet \leq 1}$ over X ,

$$X_{\bullet \leq 1} = \left[\begin{array}{c} \xrightarrow{\pi_0} \\ X_1 \xleftarrow{\delta} X_0 \\ \xrightarrow{\pi_1} \end{array} \right],$$

which satisfies the following hypotheses.

- (i) $X_0 = \coprod_{\alpha \in I_0} X_{0,\alpha}$ with $|I_0| < \infty$ is an étale covering of X by π such that $X_{0,\alpha}$ is an affine integral scheme of finite type over k for any $\alpha \in I_0$.
- (ii) $X_1 = \coprod_{\beta \in I_1} X_{1,\beta}$ with $|I_1| < \infty$ is an étale covering of $X_0 \times_X X_0$ by $\pi_0 \times \pi_1$ such that, for any $\beta \in I_1$, $X_{1,\beta}$ is an affine scheme of finite type over k and $\pi_l(X_{1,\beta}) \subset X_{0,\alpha_l}$ ($l = 0, 1$) for some $\alpha_l \in I_0$, and $\delta^{-1}(X_{1,\beta})$ coincides with one of $X_{0,\alpha}$ for some $\alpha \in I_0$ or is empty. Note that α_l ($l = 0, 1$) and α are unique, and we allow the $X_{1,\beta}$'s to be not connected.

- (iii) The inverse image $X_{s,0,\alpha}$ of X_s in $X_{0,\alpha}$ is an SNCD over k , which is defined by the inverse image of $x_{0,\alpha,1} \cdots x_{0,\alpha,r_{l,\alpha}} = 0$ of an étale morphism $X_{0,\alpha} \rightarrow \mathbb{A}_k^{n+1} = \text{Spec } k[x_{0,\alpha,1}, \dots, x_{0,\alpha,n+1}]$ for some $0 \leq r_{0,\alpha} \leq n+1$ such that the divisor $D_{0,\alpha,j}$ defined by the inverse image of $x_{0,\alpha,j} = 0$ in $X_{0,\alpha}$ is irreducible for $1 \leq j \leq r_{0,\alpha}$ and $\cap_{j=1}^{r_{0,\alpha}} D_{0,\alpha,j} \neq \emptyset$.
- (iv) For any $\alpha, \alpha' \in I_0$, $1 \leq j \leq r_{0,\alpha}$, $1 \leq j' \leq r_{0,\alpha'}$ and $\beta \in I_1$, if $\pi_0^{-1}(D_{0,\alpha,j}) \cap X_{1,\beta}$ and $\pi_1^{-1}(D_{0,\alpha',j'}) \cap X_{1,\beta}$ have a common irreducible component, then $\pi_0^{-1}(D_{0,\alpha,j}) \cap X_{1,\beta} = \pi_1^{-1}(D_{0,\alpha',j'}) \cap X_{1,\beta}$ as schemes. For $\beta \in I_1$, the set $\{D_{1,\beta,j} \mid 1 \leq j \leq r_{1,\beta}\}$ denotes all the collection of divisors of $X_{1,\beta}$ such that each of them is a form $\pi_1^{-1}(D_{0,\alpha,j}) \cap X_{1,\beta} (\neq \emptyset)$ for some $\alpha \in I_0$ and $1 \leq j \leq r_{0,\alpha}$. Note that $D_{1,\beta,j}$ is reduced and the inverse image $X_{s,1,\beta}$ of X_s in $X_{1,\beta}$ is the union of $\{D_{1,\beta,j} \mid 1 \leq j \leq r_{1,\beta}\}$.

Such a 1-truncated simplicial scheme $X_{\bullet \leq 1}$ is a 1-truncated étale hypercovering of X and it always exists. Indeed, one can put $X_{\bullet \leq 1}$, by definition of NCD's, as follows:

- (a) X_0 is an étale covering of X satisfying the hypotheses (i), (iii).
- (b) X_1 is a disjoint sum of X_0 and a Zariski open covering $\coprod_{\beta} U_{\beta} \rightarrow X_0 \times_X X_0$ of finite type such that U_{β} is an affine scheme of finite type over k which satisfies the inclusion hypothesis in (ii).
- (c) The inverse image $U_{s,\beta}$ of X_s in U_{β} is an SNCD such that the intersection of all irreducible components of $U_{s,\beta}$ is nonempty.
- (d) For $l = 0, 1$, $\pi_l : X_l \rightarrow X_0$ is given by the identity id_{X_0} on X_0 and the composition of $U_{\beta} \rightarrow X_0 \times_X X_0$ and the natural l -th projection $X_0 \times_X X_0 \rightarrow X_0$.
- (e) $\delta : X_0 \rightarrow X_1$ is the identity onto the component X_0 of X_1 .

Let us define a simplicial scheme X_{\bullet} over X by

$$X_{\bullet} = \text{cosk}_1^X(X_{\bullet \leq 1})$$

and a log-structure M_{\bullet} on X_{\bullet} by the inverse image of the log-structure M on X . Then X_{\bullet} is an étale hypercovering of X .

We will introduce a disjoint expressions on X_m for each nonnegative integer m . The disjoint expressions of $X_0 = \coprod_{\alpha \in I_0} X_{0,\alpha}$ and $X_1 = \coprod_{\alpha \in I_1} X_{1,\alpha}$ induce a disjoint expression on the fiber product $T_m = \prod_{\xi \in \text{Ob}(\Delta[1]/[m])} X_{l_{\xi}}$ as

$$T_m = \coprod_{A=((\alpha_{\xi_i}), (\beta_{\eta_{i,j}}), (\gamma_{\zeta_i}))} T_{m,A}, \quad T_{m,A} = \prod_{0 \leq i \leq m} X_{0,\alpha_{\xi_i}} \times \prod_{0 \leq i < j \leq m} X_{1,\beta_{\eta_{i,j}}} \times \prod_{0 \leq i \leq m} X_{1,\gamma_{\zeta_i}},$$

where $A = ((\alpha_{\xi_i}), (\beta_{\eta_{i,j}}), (\gamma_{\zeta_i}))$ runs over $I_0^{m+1} \times I_1^{m(m+1)/2} \times I_1^{m+1}$, and all fiber products are taken over X . Remember that X_m is a closed subscheme of T_m via the closed and open immersion h'_1 of the diagram (23). By Lemma 4.4 we have:

Lemma 4.5. *Suppose furthermore that X is connected. Then $X_m \cap T_{m,A}$ is nonempty if and only if the following conditions on $A = ((\alpha_{\xi_i}), (\beta_{\eta_{i,j}}), (\gamma_{\zeta_i}))$ hold simultaneously:*

$$\begin{cases} \pi_0(X_{1,\beta_{\eta_{i,j}}}) \subset X_{0,\alpha_{\xi_i}} & \text{for } 0 \leq i < j \leq m \\ \pi_1(X_{1,\beta_{\eta_{i,j}}}) \subset X_{0,\alpha_{\xi_j}} & \text{for } 0 \leq i < j \leq m \\ \delta^{-1}(X_{1,\gamma_{\zeta_i}}) = X_{0,\alpha_{\xi_i}} & \text{for } 0 \leq i \leq m. \end{cases}$$

In particular we have the disjoint union

$$X_m = \coprod_A (X_m \cap T_{m,A}),$$

where $A = ((\alpha_{\xi_i}), (\beta_{\eta_{i,j}}), (\gamma_{\zeta_i}))$ runs over the set of indices satisfying the conditions above.

For a general X , the lemma above holds on each connected component. We define a disjoint expression on X_m ,

$$X_m = \bigsqcup_{\alpha \in I_m} X_{m,\alpha},$$

by the induced disjoint expression from that of T_m . Note that $X_{m,\alpha}$ is not connected in general. The new disjoint expressions of X_0 and X_1 are the same with the original ones via the natural projection $T_m \rightarrow X_{\text{id}[m]} = X_m$ ($m = 0, 1$).

Lemma 4.6. (1) Let $\xi : [l] \rightarrow [m]$ be a morphism of Δ . For $\beta \in I_m$, there is a unique element $\alpha(\xi, \beta) \in I_l$ such that $\pi_\xi(X_{m,\beta}) \subset X_{l,\alpha(\xi,\beta)}$.

(2) Let $\xi : [l_\xi] \rightarrow [m], \eta : [l_\eta] \rightarrow [m]$ be objects of $\Delta[1]/[m]$. For $1 \leq j \leq r_{l_\xi, \alpha(\xi, \beta)}$ and $1 \leq j' \leq r_{l_\eta, \alpha(\eta, \beta)}$, if $\pi_\xi^{-1}(D_{l_\xi, \alpha(\xi, \beta), j}) \cap X_{m, \beta}$ and $\pi_\eta^{-1}(D_{l_\eta, \alpha(\eta, \beta), j'}) \cap X_{m, \beta}$ have a common irreducible component, then $\pi_\xi^{-1}(D_{l_\xi, \alpha(\xi, \beta), j}) \cap X_{m, \beta} = \pi_\eta^{-1}(D_{l_\eta, \alpha(\eta, \beta), j'}) \cap X_{m, \beta}$ as schemes.

Proof. (1) The assertion follows from our hypotheses: in particular (ii).

(2) It is sufficient to prove the assertion in the case where $l_\xi = l_\eta = 0$ since all divisors in the level of X_1 comes from those of X_0 by the hypothesis (iv). Suppose that $l_\xi = l_\eta = 0$ and $\xi(0) < \eta(0)$. There exists a morphism $\rho : [1] \rightarrow [m]$ such that $\pi_0 \circ \pi_\rho = \pi_\xi$ and $\pi_1 \circ \pi_\rho = \pi_\eta$. Then the assertion follows from the hypothesis (iv). \square

We denote the collection $\{D_{m,\beta,j} \mid 1 \leq j \leq r_{m,\beta}\}$ of reduced divisors of $X_{m,\beta}$ such that each of them is a form $\pi_\xi^{-1}(D_{l_\xi, \alpha(\xi, \beta), j'}) \cap X_{m, \beta} (\neq \emptyset)$ for some $\xi \in \Delta[1]/[m]$ and $1 \leq j' \leq r_{l_\xi, \alpha}$. Then the inverse image $X_{s,m,\beta}$ of X_s in $X_{m,\beta}$ is a union of $\{D_{m,\beta,j} \mid 1 \leq j \leq r_{m,\beta}\}$.

Let us now construct an embedding system $i_\bullet : X_\bullet \rightarrow \mathcal{Q}_\bullet$. For $l = 0, 1$, we fix an affine smooth formal scheme $\mathcal{R}_l = \coprod_{\alpha \in I_l} \mathcal{R}_{l,\alpha}$ separated of finite type over \mathcal{V} with an SNCD $\mathcal{E}_l = \coprod_{\alpha \in I_l} \mathcal{E}_{l,\alpha}$ relatively over \mathcal{V} , which fits into the commutative diagram over \mathcal{V} for each $\alpha \in I_l$:

$$\begin{array}{ccc} X_{l,\alpha} & \xrightarrow{i_{l,\alpha}} & \mathcal{R}_{l,\alpha} \\ \downarrow & & \downarrow \\ & & \widehat{\mathbb{A}}_{\mathcal{V}}^{s_{l,\alpha}} = \text{Spf } \mathcal{V}[y_{l,\alpha,1}, \dots, y_{l,\alpha,s_{l,\alpha}}] \\ \downarrow & & \downarrow \\ \text{Spec } k[x_{l,\alpha,1}, \dots, x_{l,\alpha,n+1}] & \longrightarrow & \widehat{\mathbb{A}}_{\mathcal{V}}^{r_{l,\alpha}} = \text{Spf } \mathcal{V}[y_{l,\alpha,1}, \dots, y_{l,\alpha,r_{l,\alpha}}] \end{array}$$

(\widehat{R} denotes a p -adic completion of R) satisfying the following hypotheses.

- (I) $i_{l,\alpha} : X_{l,\alpha} \rightarrow \mathcal{R}_{l,\alpha}$ is a closed immersion.
- (II) The left vertical arrow is the map in the hypothesis (iii) for $l = 0$ and in case $l = 1$ this comes from hypotheses (ii), (iii) and (iv).
- (III) The first right vertical arrow is smooth and the inverse image of $y_{l,\alpha,1} \cdots y_{l,\alpha,s_{l,\alpha}} = 0$ is the SNCD $\mathcal{E}_{l,\alpha} = \cup_{j=1}^{s_{l,\alpha}} \mathcal{E}_{l,\alpha,j}$, and the second right vertical arrow is the natural projection for $r_{l,\alpha} \leq s_{l,\alpha}$.
- (IV) The bottom arrow is defined by $y_{l,\alpha,j} \mapsto x_{l,\alpha,j}$ for $1 \leq j \leq r_{l,\alpha}$.
- (V) The inverse image $i_{l,\alpha}^{-1} \mathcal{E}_{l,\alpha,j}$ for $1 \leq j \leq s_{l,\alpha}$ is a sum of $D_{l,\alpha,j'}$'s with multiplicities.
- (VI) There exists a Frobenius $\sigma_{\mathcal{R}_{l,\alpha}}$ on $\mathcal{R}_{l,\alpha}$ which is an extension of the Frobenius σ on \mathcal{V} such that $\sigma_{\mathcal{R}_{l,\alpha}}(y_{l,\alpha,j}) = y_{l,\alpha,j}^p$ for any $1 \leq j \leq s_{l,\alpha}$.

Indeed, such formal schemes \mathcal{R}_l ($l = 0, 1$) exist by our hypotheses on X_0 and X_1 .

Remark 4.7. Strictly speaking the hypotheses above are more general than those we would need (and we will construct) for the aim of the present article: in fact we will be able to have $s_{l,\alpha} = r_{l,\alpha}$ and hypothesis (V) will be automatically satisfied. But we think that such hypotheses will be the correct ones if we seek a functoriality behavior to the results of our constructions.

We define a log-structure of \mathcal{R}_l which is induced by the SNCD \mathcal{E}_l . This log-structure has a local chart $L_{l,\alpha} = \mathbb{N}^{s_{l,\alpha}}$ defined by $1_{l,\alpha,j} \mapsto y_{l,\alpha,j}$, where $1_{l,\alpha,j}$ is the 1 of the j -th component of $L_{l,\alpha}$. Then $(\mathcal{R}_l, L_l^a) = \coprod_{\alpha \in I_l} (\mathcal{R}_{l,\alpha}, L_{l,\alpha}^a)$ is log-smooth over \mathcal{V} endowed with the trivial log-structure and the underlying morphism of formal schemes induces a closed immersion $(X_l, M_l) \rightarrow (R_l, L_l^a)$ of log-schemes. Here $L_{l,\alpha}^a$ means the log-structure associated to the pre-log-structure $L_{l,\alpha}$. Moreover, (\mathcal{R}_l, L_l^a) admits a lift of Frobenius $\sigma_{\mathcal{R}_l} = \coprod_{\alpha \in I_l} \sigma_{\mathcal{R}_{l,\alpha}}$.

We define a simplicial formal log-scheme \mathcal{Q}_\bullet by

$$(24) \quad (\mathcal{Q}_\bullet, \mathcal{L}_\bullet) = \Gamma_0^\mathcal{V}((\mathcal{R}_0, L_0^a))^{\leq 0} \times_{\text{Spf } \mathcal{V}} \Gamma_1^\mathcal{V}((\mathcal{R}_1, L_1^a))^{\leq 1} = \left(\prod_{\xi \in \text{Ob}(\Delta[1]/[\bullet])} (\mathcal{R}_{l_\xi}, L_{l_\xi}^a) \right).$$

as formal log-schemes over \mathcal{V} (see the definition and properties of the Γ -construction in [CT03] 11.2 and [TS04] 7.3.). Then we have a closed immersion

$$i_\bullet : (X_\bullet, M_\bullet) \rightarrow (\mathcal{Q}_\bullet, \mathcal{L}_\bullet)$$

of formal log-schemes over \mathcal{V} by [CT03], 11.2.4 and 11.2.7. Since the index set of products are same in (22) and (24), the fiber product decomposition will induce a decomposition

$$i_m = \coprod_{\alpha \in I_m} i_{m,\alpha} : \coprod_{\alpha \in I_m} (X_{m,\alpha}, M_{m,\alpha}) \rightarrow \coprod_{\alpha \in I_m} (\mathcal{Q}_{m,\alpha}, \mathcal{L}_{m,\alpha})$$

of closed immersions for each m and they form a map of simplicial formal log-schemes. (Note that the components of $(\mathcal{Q}_m, \mathcal{L}_m)$ which have no images from (X_m, M_m) can be omitted.) By the product construction in (24), $(\mathcal{Q}_\bullet, \mathcal{L}_\bullet)$ is log-smooth over \mathcal{V} and the underlying simplicial formal scheme \mathcal{Q}_\bullet is smooth. More precisely, each log-structure $\mathcal{L}_{m,\alpha}$ of $\mathcal{Q}_{m,\alpha}$ has a chart $L_{m,\alpha}$ which is a product of some of $L_{l,\beta}$ ($l = 0, 1$ and $\beta \in I_l$) and is isomorphic to $\mathbb{N}^{s_{m,\alpha}}$ with $r_{m,\alpha} \leq s_{m,\alpha}$ such that, by reordering the generators of $L_{m,\alpha}$, $1_{m,\alpha,j}$ goes to a generator of $D_{m,\alpha,j}$ in $X_{m,\alpha}$ for $1 \leq j \leq r_{m,\alpha}$ and $1_{m,\alpha,j}$ defines a sum of $D_{m,\alpha,j'}$ with multiplicities for $r_{m,\alpha} < j \leq s_{m,\alpha}$ in $X_{m,\alpha}$ by Lemma 4.6. In addition $\{L_m = \prod_{\alpha \in I_m} L_{m,\alpha} \mid m \geq 0\}$ forms a co-simplicial monoid L_\bullet by our construction of $(\mathcal{Q}_\bullet, \mathcal{L}_\bullet)$. The Frobenius endomorphisms $\sigma_{\mathcal{R}_0}$ and $\sigma_{\mathcal{R}_1}$ induce a Frobenius σ_\bullet on $(\mathcal{Q}_\bullet, \mathcal{L}_\bullet)$ such that it acts by the multiplication by p on L_\bullet .

Let us define a monoid $L_{m,\alpha}^{\text{ex}}$ in the associated group $L_{m,\alpha}^{\text{gr}}$ by

$$L_{m,\alpha}^{\text{ex}} = L_{m,\alpha} \left[\pm(1_{m,\alpha,j} - \sum_{j'=1}^{r_{m,\alpha}} \gamma_{j'} 1_{m,\alpha,j'}) \mid \begin{array}{l} r_{m,\alpha} < j \leq s_{m,\alpha}, \text{ the divisor defined by the image} \\ \text{of } y_j \text{ in } X_{m,\alpha} \text{ is } \sum_{j'=1}^{r_{m,\alpha}} \gamma_{j'} D_{m,\alpha,j'}. \end{array} \right].$$

Lemma 4.8. (1) *The composite map $L_{m,\alpha} \rightarrow \Gamma(\mathcal{Q}_{m,\alpha}, \mathcal{Q}_{m,\alpha}) \rightarrow \Gamma(X_{m,\alpha}, X_{m,\alpha})$ of monoids uniquely factors as*

$$L_{m,\alpha} \rightarrow L_{m,\alpha}^{\text{ex}} \rightarrow \Gamma(X_{m,\alpha}, X_{m,\alpha}).$$

(2) *The log-structure on $X_{m,\alpha}$ associated to pre-log-structure $L_{m,\alpha}^{\text{ex}} \rightarrow \Gamma(X_{m,\alpha}, X_{m,\alpha})$ is isomorphic to $M_{m,\alpha}$.*

(3) *The collection $\{L_m^{\text{ex}} = \prod_{\alpha \in I_m} L_{m,\alpha}^{\text{ex}} \mid m \geq 0\}$ forms a co-simplicial monoid L_\bullet^{ex} which is induced by the co-simplicial structure of M_\bullet .*

Proof. (1) If a divisor defined by the image of y_j in $X_{m,\alpha}$ is $\sum_{j'=1}^{r_{m,\alpha}} \gamma_{j'} D_{m,\alpha,j'}$, then there is a unit u of $\Gamma(X_{m,\alpha}, X_{m,\alpha})$ such that the images of y_j and $u \prod_{j'=1}^{r_{m,\alpha}} y_{j'}^{\gamma_{j'}}$ coincide with each other in $\Gamma(X_{m,\alpha}, X_{m,\alpha})$. Hence $\pm(1_{m,\alpha,j} - \sum_{j'=1}^{r_{m,\alpha}} \gamma_{j'} 1_{m,\alpha,j'})$ goes to $u^{\pm 1}$. Therefore there exists a desired factorization.

(2) Since $X_{s,m,\alpha} = \cup_{j=1}^{r_{m,\alpha}} D_{m,\alpha,j}$ is a SNCD of $X_{m,\alpha}$, the natural morphism $L_{m,\alpha} \rightarrow M_{m,\alpha}$ is surjective. The injectivity follows from Lemma 4.6.

(3) The assertion follows from our construction. \square

Let us define a simplicial log-scheme $(\mathcal{Q}_\bullet^{\text{ex}}, \mathcal{M}_\bullet)$ by

$$\mathcal{Q}_\bullet^{\text{ex}} = \mathcal{Q}_\bullet \times_{\text{Spf } \mathbb{Z}[L_\bullet]} \text{Spf } \mathbb{Z}[L_\bullet^{\text{ex}}]$$

and the log-structure \mathcal{M}_\bullet associated to the natural pre-log-structure $L_\bullet^{\text{ex}} \rightarrow \Gamma(\mathcal{Q}_\bullet^{\text{ex}}, \mathcal{Q}_\bullet^{\text{ex}})$.

Lemma 4.9. *The closed immersion $i_\bullet : (X_\bullet, M_\bullet) \rightarrow (Q_\bullet, \mathcal{L}_\bullet)$ factors as*

$$(X_\bullet, M_\bullet) \xrightarrow{i_\bullet^{\text{ex}}} (Q_\bullet^{\text{ex}}, \mathcal{M}_\bullet) \xrightarrow{h_\bullet} (Q_\bullet, \mathcal{L}_\bullet)$$

simplicial fine and saturated formal log-schemes over \mathcal{V} endowed with the trivial log-structure such that i_\bullet^{ex} is an exact closed immersion and h_\bullet is log-étale. Moreover, the underlying formal scheme Q_m^{ex} is smooth over \mathcal{V} for any m .

Proof. i_\bullet^{ex} is an exact closed immersion by Lemma 4.8 (2). Since $(L_\bullet^{\text{ex}})^{\text{gr}} = L_\bullet^{\text{gr}}$, h_m is log-étale. By the construction we have

$$\Gamma(Q_{m,\alpha}^{\text{ex}}, Q_{m,\alpha}^{\text{ex}}) = \Gamma(Q_{m,\alpha}, Q_{m,\alpha}) \left[(y_j / \prod_{j'=1}^{r_{m,\alpha}} y_{j'}^{\gamma_{j'}})^{\pm 1} \mid r_{m,\alpha} < j \leq s_{m,\alpha} \right],$$

where γ_j 's are as in the proof of Lemma 4.8. Hence $Q_{m,\alpha}^{\text{ex}}$ is smooth over \mathcal{V} . \square

Finally, the Frobenius endomorphism σ_{Q_\bullet} on $(Q_\bullet, \mathcal{L}_\bullet)$ can extend to the Frobenius σ_\bullet on $(Q_m^{\text{ex}}, \mathcal{M}_\bullet)$ such that σ_\bullet acts by multiplication by p on L_\bullet^{ex} .

This completes our proof of Proposition 4.3. \square

Remark 4.10. (1) We could have performed a similar construction for an étale hypercovering coming from a truncated one of any level (not only of level 1). For a general étale hypercovering, one may not have an embedding system of simplicial formal schemes. One may only have a truncated system. We observe that an alternative methods for the proofs of the present article could have used truncated systems via the limit arguments in [TS04] 7.5.

(2) We say that the NCD X_s of X has *self-intersections* if there are some $\alpha \in I_0$ and $j \neq j'$ such that the images of the generic points of $D_{0,\alpha,j}$ and $D_{0,\alpha,j'}$ by the étale covering $\pi : X_0 \rightarrow X$ (as in the hypotheses (i), (iii)) are the same. If the NCD X_s of X does not have self-intersections, then one can take an étale covering X_0 of X satisfying the hypotheses (i), (iii), and put $X_1 = X_0 \times_X X_0$, which is given by $X_{0,\alpha} \times_X X_{0,\alpha'} (\alpha, \alpha' \in I_0)$, with natural projections π_0, π_1 and the diagonal morphism δ . Then this 1-truncated simplicial scheme of X also satisfies the hypotheses (ii), (iv). In this case our (X_\bullet, M_\bullet) is a Čech hypercovering of (X_0, M_0) over (X, M) . Hence, if one takes a smooth lift (Q_0, L_0^a) of (X_0, M_0) , then from the Čech diagram $(Q_\bullet, \mathcal{L}_\bullet)$ of (Q_0, L_0^a) over \mathcal{V} , one can obtain a similar exactification $(Q_\bullet^{\text{ex}}, \mathcal{M}_\bullet)$.

We fix a good embedding system $i_\bullet^{\text{ex}} : (X_\bullet, M_\bullet) \rightarrow (Q_\bullet^{\text{ex}}, \mathcal{M}_\bullet)$ as in Proposition 4.3. Then using étale descent for rigid cohomology [CT03] 9.1.1 and forgetting the log-structure, we may write $(X_{s,\bullet})$ is the induced étale hypercovering of X_s)

$$(25) \quad H_{X_s, \text{rig}}^m(X) \cong \mathbb{R}^m \Gamma([X_\bullet]_{Q_\bullet^{\text{ex}}}, [\Omega_\bullet^{\text{ex}}]_{[X_\bullet]_{Q_\bullet^{\text{ex}}}} \rightarrow j_{[X_\bullet \setminus X_{s,\bullet}]_{Q_\bullet^{\text{ex}}}}^\dagger \Omega_\bullet^{\text{ex}}]_{[X_\bullet]_{Q_\bullet^{\text{ex}}}}).$$

We are now ready to interpret the two complexes which appear in the simple complexes of the right hand side of (25). The first one is just calculating the rigid cohomology $H_{\text{rig}}^m((X, X))$ of the pair (X, X) . The second complex is nothing but the rigid cohomology $H_{\text{rig}}^m((X \setminus X_s, X))$ of the pair $(X \setminus X_s, X)$. Then we may apply Shiho's result [SH02] 2.4.4: it says that, for the smooth log-scheme (X_m, M_m) which has a Zariski type log structure, its log-convergent cohomology over \mathcal{V} endowed with the trivial log-structure coincides with the rigid cohomology of the pair $(X_m \setminus X_{s,m}, X_m)$. Note again that M_m is the log-structure of X_m induced from the SNCD $X_{s,m}$ and the trivial log locus of (X_m, M_m) is $X_m \setminus X_{s,m}$. Hence we have

$$H_{\text{rig}}^m((X \setminus X_s, X)) \cong H_{\text{log-conv}}^m((X, M)/\mathcal{V}).$$

To avoid confusion we stress the fact that $H_{\text{log-conv}}^m((X, M)/\mathcal{V})$ represent the log-convergent cohomology groups of (X, M) relative to \mathcal{V} endowed with the trivial log-structure.

To connect our construction to the log theory we will use $] - [^{\text{log}}$ to refer to log-tubes for exact or non exact closed immersions, which has already appeared in paragraph 3: for example the log-tubes of X_\bullet for a generic embedding (not exact) Q_\bullet will be indicated by $]X_\bullet[_{Q_\bullet}^{\text{log}}$. Of course for our Q_\bullet^{ex} we have

$$]X_\bullet[_{Q_\bullet}^{\text{log}} =]X_\bullet[_{Q_\bullet^{\text{ex}}},$$

where the second is the "classical" tube, since the closed immersion $i_{\bullet}^{\text{ex}} : (X_{\bullet}, M_{\bullet}) \rightarrow (Q_{\bullet}^{\text{ex}}, \mathcal{M}_{\bullet})$ is an exactification of $(X_{\bullet}, M_{\bullet}) \rightarrow (Q_{\bullet}, \mathcal{L}_{\bullet})$. Hence the log-convergent cohomology $H_{\log\text{-conv}}^m((X, M)/\mathcal{V}) = H_{\log\text{-conv}}^m((X_{\bullet}, M_{\bullet})/\mathcal{V})$ can be calculated by

$$\mathbb{R}^m \Gamma([X_{\bullet}, \Omega_{Q_{\bullet}^{\text{ex}}}^{\bullet}, \Omega_{[X_{\bullet}, Q_{\bullet}^{\text{ex}}]}^{\bullet} < \mathcal{M}_{\bullet} >]) = \mathbb{R}^m \Gamma([X_{\bullet}, \Omega_{Q_{\bullet}^{\text{ex}}}^{\bullet}, \Omega_{[X_{\bullet}, Q_{\bullet}^{\text{ex}}]}^{\bullet} < \mathcal{M}_{\bullet} >]),$$

where $\Omega_{[X_{\bullet}, Q_{\bullet}^{\text{ex}}]}^{\bullet} < \mathcal{M}_{\bullet} >$ indicates the log differential of the generic fiber of $(Q_{\bullet}^{\text{ex}}, \mathcal{M}_{\bullet})$. We may then write (25) as

$$H_{X_s, \text{rig}}^m(X) \cong \mathbb{R}^m \Gamma([X_{\bullet}, \Omega_{Q_{\bullet}^{\text{ex}}}^{\bullet}, [\Omega_{[X_{\bullet}, Q_{\bullet}^{\text{ex}}]}^{\bullet} \rightarrow \Omega_{[X_{\bullet}, Q_{\bullet}^{\text{ex}}]}^{\bullet} < \mathcal{M}_{\bullet} >]])$$

Now we would like to continue our interpretation. As a matter of fact we are going to use another *exact embedding system* to calculate the cohomology of the complex $[\Omega_{[X_{\bullet}, Q_{\bullet}^{\text{ex}}]}^{\bullet} \rightarrow \Omega_{[X_{\bullet}, Q_{\bullet}^{\text{ex}}]}^{\bullet} < \mathcal{M}_{\bullet} >]$. In fact we are going to use a new good embedding system $\tilde{Q}_{\bullet}^{\text{ex}}$ (exact, smooth as before) which admits a map to $\mathcal{C}_{\mathcal{V}}$ and this map is log-smooth over $(\mathcal{C}_{\mathcal{V}}, \mathcal{N})$.

Proposition 4.11. *In the previous notation, it is possible to find a good embedding system $X_{\bullet} \rightarrow \tilde{Q}_{\bullet}^{\text{ex}}$ which fits in the diagram*

$$\begin{array}{ccc} (X_{\bullet}, M_{\bullet}) & \xrightarrow{\tilde{i}_{\bullet}^{\text{ex}}} & (\tilde{Q}_{\bullet}^{\text{ex}}, \tilde{\mathcal{M}}_{\bullet}) \\ \downarrow & & \downarrow \\ (C, s) & \longrightarrow & (\mathcal{C}_{\mathcal{V}}, \mathcal{N}) \end{array}$$

where the horizontal maps are exact closed immersions and $(\tilde{Q}_{\bullet}^{\text{ex}}, \tilde{\mathcal{M}}_{\bullet})$ is log-smooth over $(\mathcal{C}_{\mathcal{V}}, \mathcal{N})$ and the underlying formal scheme $\tilde{Q}_{\bullet}^{\text{ex}}$ is smooth and separated of finite type over \mathcal{V} . Moreover $(\tilde{Q}_{\bullet}^{\text{ex}}, \tilde{\mathcal{M}}_{\bullet})$ admits a lift $\tilde{\sigma}_{\bullet}$ of Frobenius which is compatible with the Frobenius $\sigma_{\mathcal{C}_{\mathcal{V}}}$ on $(\mathcal{C}_{\mathcal{V}}, \mathcal{N})$.

Proof. Let us keep the notations as in the proof of Proposition 4.3 and let t be a section of a coordinate of \hat{s} in $\mathcal{C}_{\mathcal{V}}$. Then the log-structure \mathcal{N} on $\mathcal{C}_{\mathcal{V}}$ is obtained by the pre log-structure $\mathbb{N} \rightarrow \Gamma(\mathcal{C}_{\mathcal{V}}, \mathcal{O}_{\mathcal{C}_{\mathcal{V}}})$ given by $1 \mapsto t$. We fix a Frobenius $\sigma_{\mathcal{C}_{\mathcal{V}}}$ on $\mathcal{C}_{\mathcal{V}}$, which is compatible with the Frobenius σ on \mathcal{V} , such that $\sigma_{\mathcal{C}_{\mathcal{V}}}(t) = t^p$.

If we consider the tensor product $(Q_{\bullet}^{\text{ex}}, \mathcal{M}_{\bullet}) \times (\mathcal{C}_{\mathcal{V}}, \mathcal{N})$ over \mathcal{V} endowed with trivial log-structure, then the natural morphism $(X_{\bullet}, M_{\bullet}) \rightarrow (Q_{\bullet}^{\text{ex}}, \mathcal{M}_{\bullet}) \times (\mathcal{C}_{\mathcal{V}}, \mathcal{N})$ is a closed immersion, but is not exact. We will need to modify it in order to get an exact one. Note that the log-structure $(Q_{\bullet}^{\text{ex}}, \mathcal{M}_{\bullet}) \times (\mathcal{C}_{\mathcal{V}}, \mathcal{N})$ is the associated log-structure to the monoid $L_{\bullet}^{\text{ex}} \times \mathbb{N}$.

We define a co-simplicial monoid $\tilde{L}_{\bullet}^{\text{ex}}$ by

$$\begin{aligned} \tilde{L}_m^{\text{ex}} &= \prod_{\alpha \in I_m} \tilde{L}_{m, \alpha}^{\text{ex}}, \\ \tilde{L}_{m, \alpha}^{\text{ex}} &= L_{m, \alpha}^{\text{ex}} \times \mathbb{N}[\pm((1_{m, \alpha, 1} + \dots + 1_{m, \alpha, r_{m, \alpha}}) - 1)] \text{ in } (L_{m, \alpha}^{\text{ex}} \times \mathbb{N})^{\text{gr}}, \end{aligned}$$

where 1 is the generator of the last component \mathbb{N} . Indeed, the collection $\{\tilde{L}_m^{\text{ex}} \mid m \geq 0\}$ forms a co-simplicial monoid since $x_{m, \alpha, 1} x_{m, \alpha, 2} \dots x_{m, \alpha, r_{m, \alpha}} = ut$ for a unit u on $X_{m, \alpha}$ and by the fact that the closed immersion $i_{\bullet}^{\text{ex}} : (X_{\bullet}, M_{\bullet}) \rightarrow (Q_{\bullet}^{\text{ex}}, \mathcal{M}_{\bullet})$ is exact. Moreover, the associated log-structure on X_{\bullet} linked to the natural homomorphism $\tilde{L}_{\bullet}^{\text{ex}} \rightarrow \Gamma(X_{\bullet}, \mathcal{O}_{X_{\bullet}})$ of monoids is the given log-structure M_{\bullet} .

We define a simplicial formal log-scheme $(\tilde{Q}_{\bullet}^{\text{ex}}, \tilde{\mathcal{M}}_{\bullet})$ over \mathcal{V} by

$$\tilde{Q}_{\bullet}^{\text{ex}} = (Q_{\bullet}^{\text{ex}} \times \mathcal{C}_{\mathcal{V}}) \times_{\text{Spf } \mathbb{Z}[L_{\bullet}^{\text{ex}} \times \mathbb{N}]} \text{Spf } \mathbb{Z}[\tilde{L}_{\bullet}^{\text{ex}}]$$

and $\tilde{\mathcal{M}}_{\bullet}$ is the log-structure associated to the natural homomorphism $\tilde{L}_{\bullet}^{\text{ex}} \rightarrow \Gamma(\tilde{Q}_{\bullet}^{\text{ex}}, \mathcal{O}_{\tilde{Q}_{\bullet}^{\text{ex}}})$. By the similar proof of Lemma 4.9 we have

Lemma 4.12. (1) *There is a natural commutative diagram*

$$\begin{array}{ccc}
 & & (\widetilde{Q}_\bullet^{\text{ex}}, \widetilde{\mathcal{M}}_\bullet) \\
 & \nearrow & \downarrow \\
 (X_\bullet, M_\bullet) & \rightarrow & (Q_\bullet^{\text{ex}}, \mathcal{L}_\bullet) \times (\mathcal{C}_V, \mathcal{N}) \\
 \downarrow & & \downarrow \text{2-nd proj.} \\
 (C, s) & \rightarrow & (\mathcal{C}_V, \mathcal{N})
 \end{array}$$

of formal log-schemes over \mathcal{V} such that $(X_\bullet, M_\bullet) \rightarrow (\widetilde{Q}_\bullet^{\text{ex}}, \widetilde{\mathcal{M}}_\bullet)$ is an exact closed immersion and $(\widetilde{Q}_\bullet^{\text{ex}}, \widetilde{\mathcal{M}}_\bullet) \rightarrow (Q_\bullet^{\text{ex}}, \mathcal{L}_\bullet) \times (\mathcal{C}_V, \mathcal{N})$ is log-étale. In particular, each level of $(\widetilde{Q}_\bullet^{\text{ex}}, \widetilde{\mathcal{M}}_\bullet)$ is log-smooth over $(\mathcal{C}_V, \mathcal{N})$.

(2) *The underlying simplicial formal scheme $\widetilde{Q}_\bullet^{\text{ex}}$ is smooth over \mathcal{V} .*

(3) *The morphism $\sigma_\bullet \times \sigma_{\mathcal{C}_V}$ on $(Q_\bullet^{\text{ex}}, \mathcal{M}_\bullet) \times (\mathcal{C}_V, \mathcal{N})$ can be extended to a morphism $\widetilde{\sigma}_\bullet : (\widetilde{Q}_\bullet^{\text{ex}}, \widetilde{\mathcal{M}}_\bullet) \rightarrow (\widetilde{Q}_\bullet^{\text{ex}}, \widetilde{\mathcal{M}}_\bullet)$ such that $\widetilde{\sigma}_\bullet$ is compatible with the Frobenius σ on \mathcal{V} and acts by multiplying p on $\widetilde{L}_\bullet^{\text{ex}}$.*

Now the proof of Proposition 4.11 is complete. \square

Hence we may use the exact embedding of the previous proposition to calculate our simple complex of (25), note that the log-tubes coincide with the usual one: $]X_\bullet[_{\widetilde{Q}_\bullet^{\text{ex}}}^{\log} =]X_\bullet[_{\widetilde{Q}_\bullet^{\text{ex}}}$. This is true because the immersion is exact and we have

$$(26) \quad H_{X_s, \text{rig}}^m(X) \cong \mathbb{R}^m \Gamma(]X_\bullet[_{\widetilde{Q}_\bullet^{\text{ex}}}[\Omega_{]X_\bullet[_{\widetilde{Q}_\bullet^{\text{ex}}}}^\bullet \rightarrow \Omega_{]X_\bullet[_{\widetilde{Q}_\bullet^{\text{ex}}}}^\bullet < \widetilde{\mathcal{M}}_\bullet >]).$$

But we can also introduce an admissible covering of $]X_\bullet[_{\widetilde{Q}_\bullet^{\text{ex}}}$: this is given by the inverse image of the admissible covering of \mathcal{C}_K given by the tube of $\{s\}$ (open unit disk hence quasi-Stein) and V where V is a strict affinoid neighborhood of $C \setminus \{s\}$ in \mathcal{C}_V . The inverse image of such a covering gives an admissible covering of $]X_\bullet[_{\widetilde{Q}_\bullet^{\text{ex}}}$: $]X_{s, \bullet}[_{\widetilde{Q}_\bullet^{\text{ex}}}$ and V_\bullet which are respectively the inverse image of the tube of $\{s\}$ and of V in $]X_\bullet[_{\widetilde{Q}_\bullet^{\text{ex}}}$. In particular the open immersion $]X_{s, \bullet}[_{\widetilde{Q}_\bullet^{\text{ex}}} \rightarrow]X_\bullet[_{\widetilde{Q}_\bullet^{\text{ex}}}$ is a quasi-Stein map. The restrictions to V_\bullet of the two complexes which appear in (26) are the same: using this fact and the quasi-Stein property and because all the sheaves are coherent, we can again re-write (26). In fact, $H_{X_s, \text{rig}}^i(X)$ can be calculated as the derived functors of the global section functor on $]X_{s, \bullet}[_{\widetilde{Q}_\bullet^{\text{ex}}}$ of

$$(27) \quad [\Omega_{]X_{s, \bullet}[_{\widetilde{Q}_\bullet^{\text{ex}}}}^\bullet \rightarrow \Omega_{]X_{s, \bullet}[_{\widetilde{Q}_\bullet^{\text{ex}}}}^\bullet < \widetilde{\mathcal{M}}_\bullet >].$$

By our hypotheses X_s is proper and $\widetilde{Q}_\bullet^{\text{ex}}$ is smooth, the first complex calculates the rigid cohomology of X_s , while the second calculates the log-convergent cohomology of X_s endowed with the induced log-structure from X , i.e. the log-convergent cohomology of (X_s, M_s) with respect to the trivial log-structure on the base field, $H_{\log\text{-conv}}^m((X_s, M_s)/\mathcal{V})$.

Remark 4.13. It is tempting to give a name to the hypercohomology of the global sections functor of the previous simple complex (27), and to denote it by $H_{X_s, \text{rig}}^m(\hat{X}/K)$, where \hat{X} is the completion of X along X_s .

As a corollary of these two interpretations, we obtain the following long exact sequence of finite dimensional K -vector spaces:

$$(2) \quad \cdots \longrightarrow H_{X_s, \text{rig}}^m(X) \xrightarrow{\alpha} H_{\text{rig}}^m(X_s) \longrightarrow H_{\log\text{-conv}}^m((X_s, M_s)/\mathcal{V}) \longrightarrow \cdots.$$

This long exact sequence is compatible with Frobenius. By classical results we know that the rigid terms of such a long exact sequence have Frobenius structures (they have a structure of mixed F -isocrystals [CH98]), hence we may endow $H_{\log\text{-conv}}^m((X_s, M_s)/\mathcal{V})$ with a Frobenius structure (even if (X_s, M_s) is not log-smooth over k endowed with the trivial log-structure). Again we want to stress the fact that $H_{\log\text{-conv}}^m((X_s, M_s)/\mathcal{V})$ represents the m -th log-convergent cohomology group of (X_s, M_s) relative to the trivial log-structure on the base field k . We recall that we indicate by $H_{\log\text{-conv}}^m((X_s, M_s)/\mathcal{V}^\times)$

the log-convergent cohomology groups of (X_s, M_s) over \mathcal{V}^\times (endowed with the log-structure of \mathcal{V} associated to $\mathbb{N} \ni 1 \mapsto 0 \in \mathcal{V}$). Such cohomology groups coincide with the log-crystalline ones: $H_{\log\text{-crys}}^m((X_s, M_s)/\mathcal{V}^\times) \otimes K$ (see notation before (11)).

Remark 4.14. (1) We should remark that we could have proved the existence of (2) by just using the exact embedding system $(X_\bullet, M_\bullet) \rightarrow (Q_\bullet^{\text{ex}}, \mathcal{M}_\bullet)$. In this case, V_\bullet , of the covering above, would have been constructed as the complement of tube of $X_{s,\bullet}$ of radius $\eta < 1$.

(2) More generally if we try to deal with a general Hyodo-Kato embedding system $(X_\bullet, M_\bullet) \rightarrow (\mathcal{P}_\bullet, \mathcal{M}_\bullet)$ as in (7), then we would have replaced the complex in (26) by

$$[\Omega_{X_\bullet, [\mathcal{P}_\bullet]}^\bullet \rightarrow h_{K, \bullet, \bullet} \Omega_{X_\bullet, [\mathcal{P}_\bullet]}^\bullet \xrightarrow{\log} \mathcal{M}_\bullet >]$$

where $h_{K, \bullet, \bullet} :]X_\bullet, [\mathcal{P}_\bullet]^\log \rightarrow]X_\bullet, [\mathcal{P}_\bullet]$ is the canonical morphism. As a corollary of our local global comparison, we have an isomorphism

$$H_{X_s, \text{rig}}^m(X) \cong \mathbb{R}^m \Gamma(]X_{s, \bullet}, [\mathcal{P}_\bullet, [\Omega_{X_{s, \bullet}, [\mathcal{P}_\bullet]}^\bullet \rightarrow h_{K, s, \bullet, \bullet} \Omega_{X_{s, \bullet}, [\mathcal{P}_\bullet]}^\bullet \xrightarrow{\log} \mathcal{M}_\bullet >])$$

where $h_{K, s, \bullet, \bullet} :]X_{s, \bullet}, [\mathcal{P}_\bullet]^\log \rightarrow]X_{s, \bullet}, [\mathcal{P}_\bullet]$ is the restriction of $h_{K, \bullet, \bullet}$. However, one can not directly apply our argument of global and local comparison to this complex, for the reason that $h_{K, \bullet, \bullet}^{-1}(V_\bullet)$ is not isomorphic to V_\bullet in general. As a matter of fact, the local existence of exact embeddings is enough in order to prove the exact sequence (2). The authors, however, thought that it were worth to prove the existence of global exact embedding systems of Propositions 4.3, 4.11 for further studies.

* * *

By putting together (2) and (3) we have the sequence (as in the introduction)

$$(1) \quad \begin{aligned} \cdots &\rightarrow H_{\text{rig}}^m(X_s) \xrightarrow{\gamma} H_{\log\text{-crys}}^m((X_s, M_s)/\mathcal{V}^\times) \otimes K \xrightarrow{N_m} H_{\log\text{-crys}}^m((X_s, M_s)/\mathcal{V}^\times) \otimes K(-1) \xrightarrow{\delta} H_{X_s, \text{rig}}^{m+2}(X) \\ &\xrightarrow{\alpha} H_{\text{rig}}^{m+2}(X_s) \rightarrow \cdots \end{aligned}$$

where the maps γ and δ are defined from (2) and (3) by the composites

$$H_{\text{rig}}^m(X_s) \rightarrow H_{\log\text{-conv}}^m((X_s, M_s)/\mathcal{V}) \rightarrow H_{\log\text{-crys}}^m((X_s, M_s)/\mathcal{V}^\times) \otimes K$$

and

$$H_{\log\text{-crys}}^m((X_s, M_s)/\mathcal{V}^\times) \otimes K(-1) \rightarrow H_{\log\text{-conv}}^{m+1}((X_s, M_s)/\mathcal{V}) \rightarrow H_{X_s, \text{rig}}^{m+2}(X).$$

5 Exactness of the sequence

In this last part we would like to prove the exactness of the previous long sequence (as in the introduction)

$$(1) \quad \begin{aligned} \cdots &\rightarrow H_{\text{rig}}^m(X_s) \xrightarrow{\gamma} H_{\log\text{-crys}}^m((X_s, M_s)/\mathcal{V}^\times) \otimes K \xrightarrow{N_m} H_{\log\text{-crys}}^m((X_s, M_s)/\mathcal{V}^\times) \otimes K(-1) \xrightarrow{\delta} H_{X_s, \text{rig}}^{m+2}(X) \\ &\xrightarrow{\alpha} H_{\text{rig}}^{m+2}(X_s) \rightarrow \cdots \end{aligned}$$

and we will make the further hypothesis that the field k is finite with $q = p^a$ elements. We will show how such a result (after our translation of the topological tools in our framework) is a formal corollary of the fact that in characteristic p the monodromy filtration coincides with the weight filtration. We recall that in the third paragraph we had the following interpretation (12):

$$(\mathbb{R}^m f_{(X, M)/(\mathcal{C}_{\mathcal{V}}, \mathcal{N}), \text{an}^*(\text{an}, X, K)}]_{s[\mathcal{C}_{\mathcal{V}}]} \delta_K \cong H_{\log\text{-crys}}^m((X_s, M_s)/\mathcal{V}^\times) \otimes K,$$

where the tube $]s[_{\mathcal{C}_V}$ is isomorphic, by a choice of the lift \hat{s} of s , to the open unit disk with a parameter t corresponding to \hat{s}_K . The relative log-analytic cohomology can be represented by the trivialization (13):

$$(\mathbb{R}^m f_{(X,M)/(\mathcal{C}_V, \mathcal{N})}]s[_{\mathcal{C}_V, \text{an}^*(\text{an}, X, K)}, \nabla, \varphi_m) \cong (V_m \otimes]s[_{\mathcal{C}_V}, N_m \otimes 1 + 1 \otimes d, F_m \otimes \sigma_{\mathcal{C}_V}).$$

where V_m is a K -vector space endowed with a monodromy endomorphism N_m and a Frobenius structure F_m such that $qF_m^a N_m = N_m F_m^a$. Moreover, V_m is isomorphic to $H_{\log\text{-crys}}^m((X_s, M_s)/\mathcal{V}^\times) \otimes K$ by Corollary 3.10.

Associated with a nilpotent operator we have a *monodromy* filtration on V_m . We will indicate it by \mathcal{M} and the steps by \mathcal{M}_j , $j \in \mathbb{Z}$. Moreover the Frobenius F_m^a induces a filtration called the *weight* filtration. In this setting we have the equivalence between the two filtrations along the line of Crew's proof [CR98] 10.8:

Theorem 5.1. *Under the previous hypotheses, the relative log-analytic cohomology sheaf*

$$\mathbb{R}^m f_{(X,M)/(\mathcal{C}_V, \mathcal{N}), \text{an}^*(\text{an}, X, K)}$$

on $\mathcal{C}_K \setminus]s[_{\mathcal{C}_V}$ is pure of weight m . Moreover $\mathbb{R}^m f_{(X,M)/(\mathcal{C}_V, \mathcal{N}), \text{an}^(\text{an}, X, K)}$ on $]s[_{\mathcal{C}_V}$ is unipotent and, for each j , the graded part for the monodromy filtration, $gr_j^{\mathcal{M}} V_m$, is pure of weight $m + j$ for the Frobenius action.*

Proof. In view of our hypotheses the family $X \rightarrow C$ was given by a proper and log-smooth map with only one (classically) singular fiber at s . Hence $\mathbb{R}^m f_{(X,M)/(\mathcal{C}_V, \mathcal{N}), \text{an}^*(\text{an}, X, K)}$ calculates the cohomology of the proper and smooth fibers except X_s . Hence they are pure for the Frobenius structure. Now we can apply Crew's theorem [Ibid.]. \square

This gives an equivalence between monodromy and weight filtrations (up to a shift).

Corollary 5.2. *On $H_{\log\text{-crys}}^m((X_s, M_s)/\mathcal{V}^\times) \otimes K$ the $\ker N_m$ has (Frobenius) weights less than or equal to m . While $H_{\log\text{-crys}}^m((X_s, M_s)/\mathcal{V}^\times) \otimes K(-1)/\text{Im } N_m$ has (Frobenius) weights greater than or equal to $m + 2$.*

Proof. By Deligne [DE80] 1.6, we know that N_m is injective on $V_m/\mathcal{M}_0 \rightarrow V_m/\mathcal{M}_{-2}(-1)$, hence $\ker N_m \subset \mathcal{M}_0$. On the other hand for each $j < 0$, $N_m \mathcal{M}_{j+2} = \mathcal{M}_j(-1)$: hence $\mathcal{M}_{-1}(-1) \subset \text{Im } N_m$. We conclude that $V_m(-1)/\text{Im } N_m$ has weights bigger than or equal to $m + 2$. \square

Corollary 5.3. *The Frobenius induces a structure of mixed F^a -isocrystal on $H_{\log\text{-conv}}^m((X_s, M_s)/\mathcal{V})$ ([CH98]). Moreover we have a direct sum decomposition with respect to the weights*

$$H_{\log\text{-conv}}^m((X_s, M_s)/\mathcal{V}) = H_{\log\text{-conv}}^m((X_s, M_s)/\mathcal{V})^{\leq m} \oplus H_{\log\text{-conv}}^m((X_s, M_s)/\mathcal{V})^{> m}.$$

Proof. The cohomological groups $H_{\log\text{-conv}}^m((X_s, M_s)/\mathcal{V})$ sit in two long exact sequences (2) and (3) both compatible with Frobenius: hence they are mixed as F^a -isocrystals. In (2), by the theory of "classical" rigid cohomology we know that $H_{\text{rig}}^m(X_s)$ has weights $\leq m$ ([CL98] 2.2) while $H_{X_s, \text{rig}}^{m+1}(X)$ has weights strictly bigger than m ([CH98], 2.3): then one can insert $H_{\log\text{-conv}}^m((X_s, M_s)/\mathcal{V})$ in a short exact sequence where the first non trivial term has weights $\leq m$ (in fact it is a quotient of $H_{\text{rig}}^m(X_s)$), while the last non trivial term has weights $> m$ (because it is a sub F^a -isocrystal of $H_{X_s, \text{rig}}^{m+1}(X)$). But, using (3), we can insert $H_{\log\text{-conv}}^m((X_s, M_s)/\mathcal{V})$ in another short exact sequence: this time the first non trivial term is the quotient $H_{\log\text{-crys}}^{m-1}((X_s, M_s)/\mathcal{V}^\times) \otimes K(-1)/\text{Im } N_{m-1}$ which has weights $> m$ by Corollary 5.2 and the last term is $\ker N_m$ on $H_{\log\text{-crys}}^m((X_s, M_s)/\mathcal{V}^\times) \otimes K$ which has weights $\leq m$ (again by Corollary 5.2). \square

We are now ready for the last step in the proof of exactness of the Clemens-Schmid sequence. As we said, the part of the long exact sequence (2)

$$\cdots \rightarrow H_{\text{rig}}^m(X_s) \rightarrow H_{\log\text{-conv}}^m((X_s, M_s)/\mathcal{V}) \rightarrow H_{X_s, \text{rig}}^{m+1}(X) \rightarrow \cdots$$

is compatible with the Frobenius structure, hence it gives a surjection

$$H_{\text{rig}}^m(X_s) = H_{\text{rig}}^m(X_s)^{\leq m} \rightarrow H_{\log\text{-conv}}^m((X_s, M_s)/\mathcal{V})^{\leq m}.$$

We then have $\text{Im}(H_{\text{rig}}^m(X_s) \rightarrow H_{\log\text{-conv}}^m((X_s, M_s)/\mathcal{V})) = H_{\log\text{-conv}}^m((X_s, M_s)/\mathcal{V})^{\leq m}$ and of course it is contained in $\ker N_m$ according to the long exact sequence (3). Since $\ker N_m$ has weights at most m and the Coker N_{m-1} has weights $\geq m+1$, then the kernel is exactly isomorphic to $H_{\log\text{-conv}}^m((X_s, M_s)/\mathcal{V})^{\leq m}$ under the map in (3). Hence it is $\text{Im}(H_{\text{rig}}^m(X_s) \rightarrow H_{\log\text{-crys}}^m((X_s, M)/\mathcal{V}^\times) \otimes K)$ in γ of (1). In the second part of the sequence, we know that $\text{Coker } N_m$ has weights $\geq m+2$ (because of the Frobenius twist). And it is isomorphic to $H_{\log\text{-conv}}^{m+1}(X_s, M_s)/\mathcal{V}^{\geq m+2}$ because $\ker N_{m+1}$ has weights $\leq m+1$. Consider the long exact sequence (2)

$$H_{\log\text{-conv}}^{m+1}((X_s, M_s)/\mathcal{V}) \rightarrow H_{X_s, \text{rig}}^{m+2}(X) \rightarrow H_{\text{rig}}^{m+2}(X_s).$$

The kernel of the last map is isomorphic to $H_{\log\text{-conv}}^{m+1}((X_s, M_s)/\mathcal{V})^{\geq m+2}$ because $H_{\text{rig}}^{m+1}(X_s)$ has weights $\leq m+1$ and $H_{X_s, \text{rig}}^{m+2}(X)$ has weights $\geq m+2$.

This concludes the proof of the exactness of the Clemens-Schmid sequence.

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